VECTORS

Vectors are used to describe quantities that have magnitude and direction.

Examples:

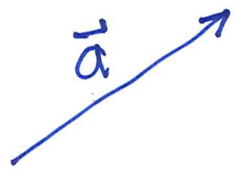
- Velocity
- acceleration
- force
- momentum

Scalars are constant

- Energy
- time
- mass

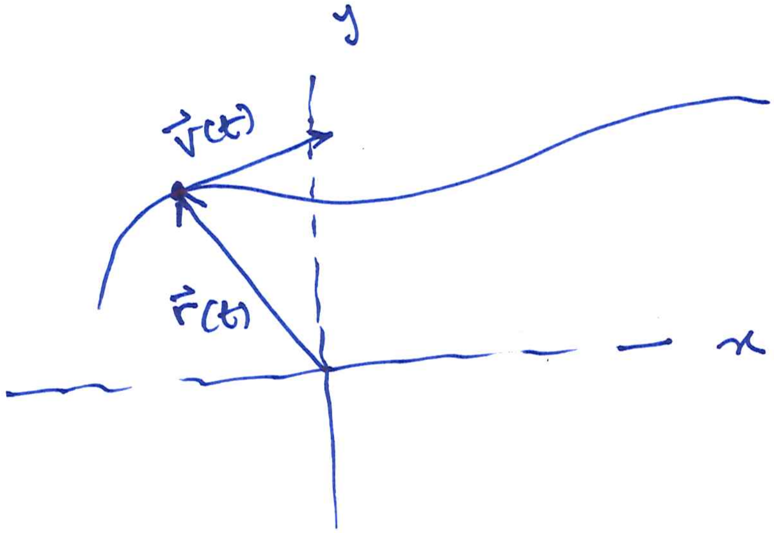
⋮

Let $\vec{a}$ be a vector



$\vec{a}$

$\hat{a}$



vector operations

① Scalar multiplication

suppose $\vec{a}$ is vector and α is a scalar

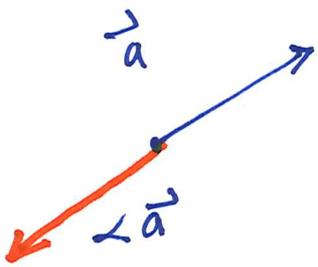
then $\alpha \vec{a}$ is a scalar multiplication.

(i) Suppose $\alpha = 1$.

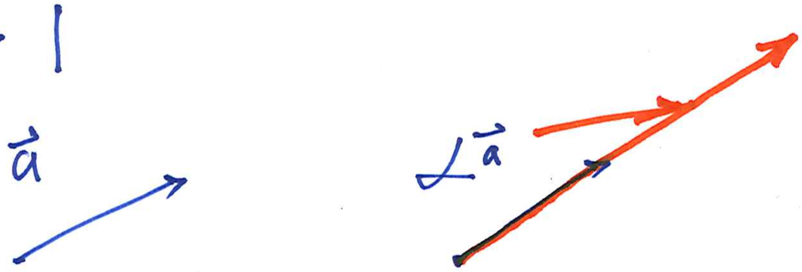
$$\alpha \vec{a} = \vec{a}$$

(ii) Suppose $\alpha = -1$

$$\alpha \vec{a} = -\vec{a}$$



(iii) $\alpha > 1$

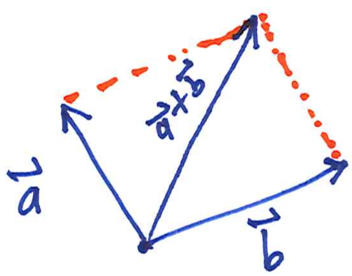


(iv) $0 < \alpha < 1$

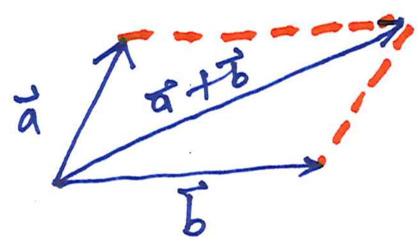
vector addition

Suppose we have vectors $\vec{a}$ and $\vec{b}$

(i)



(ii)



vector coordinates

Suppose $\vec{a}$ is vector in 2 dimension

$$\vec{a} = \underline{\underline{(a_1, a_2)}} \equiv [a_1, a_2] \equiv \langle a_1, a_2 \rangle$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are the standard basic vectors

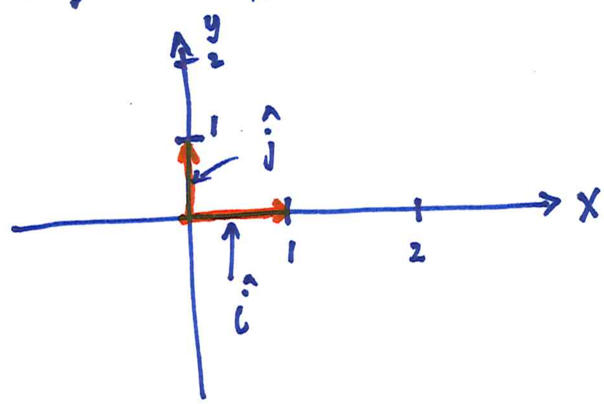
If $\vec{a}$ is 3 dimensions.

$$\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

In 2D

$\hat{i} = (1, 0)$

$\hat{j} = (0, 1)$

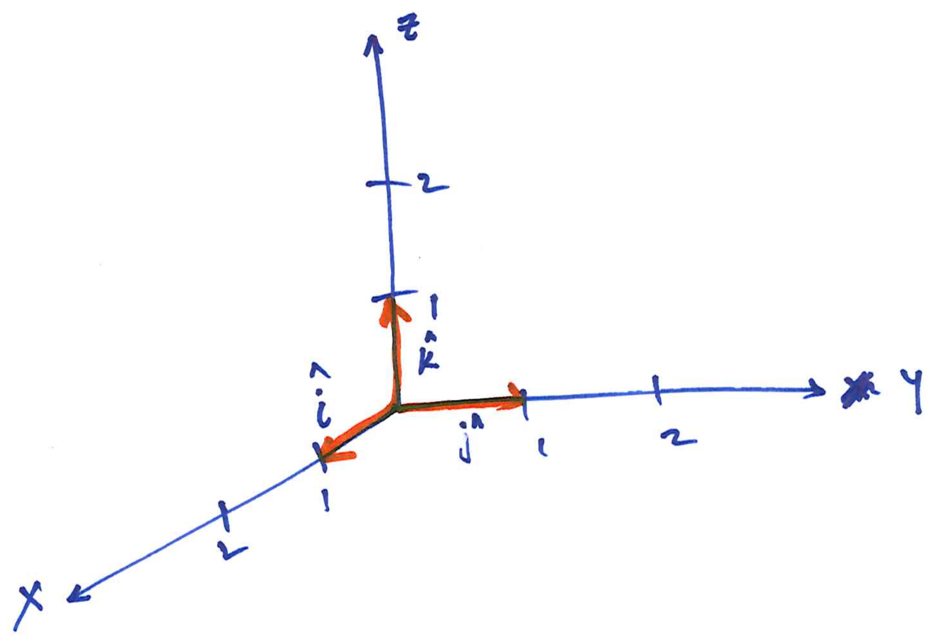


In 3 D

$\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0)$

$\hat{k} = (0, 0, 1)$



In 2D

$$\vec{a} = (a_1, a_2) = a_1 i + a_2 j$$

$$= a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$ — column vector

$$= a_1 (1, 0) + a_2 (0, 1) = (a_1, a_2)$$ — row vector

For n dimensions,

$$\vec{a} = (a_1, a_2, \text{——}, a_n)$$ n — positive integer.

Scalar multiplication

Let $\vec{a} = (a_1, a_2)$ and α be a scalar

(1) $\alpha\vec{a} = \alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

if $\vec{a}$ is a vector in n dimension

ie $\vec{a} = (a_1, a_2, \ldots, a_n)$

then $\alpha\vec{a} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$.

vector addition

If we have two vector $\vec{a}$ and $\vec{b}$

$\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$\vec{a} + \vec{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3)$

$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$\vec{a} + \vec{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$

$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

Example:

(1) Let $\vec{a} = (3, 1)$ and $\vec{b} = (-2, 3)$

(i) $\vec{a}+\vec{b} = (1, 4)$

(ii) $3\vec{a} + 5\vec{b} = 3(3, 1) + 5(-2, 3)$

$= (9, 3) + (-10, 15)$

$= (-1, 18)$.

properties of ~~vectors~~ vector addition and scalar multiplication

(I) Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors of the same dimension and $\vec{0}$ be a zero vector

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(ii) $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$

(iii) $\vec{a}+\vec{0} = \vec{a}$

(iv) $\vec{a}+(-\vec{a}) = \vec{0}$

If α and β are scalars,

(v) $\alpha(\vec{a}+\vec{b}) = \alpha\vec{a} + \alpha\vec{b}$

(vi) $(\alpha+\beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$

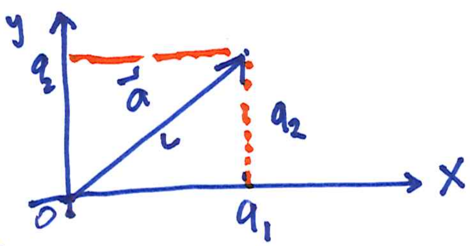
(vii) $(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$

(vii) $1\cdot\vec{a} = \vec{a}$.

Length of a vector

we denote the length of a vector by $\|\vec{a}\|$

suppose $\vec{a} = (a_1, a_2)$



using pythagoras theorem,

$$L^2 = a_1^2 + a_2^2$$

$$\|\vec{a}\|^2 = a_1^2 + a_2^2$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

If $\vec{a} = (a_1, a_2, \dots, a_n)$

Then $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

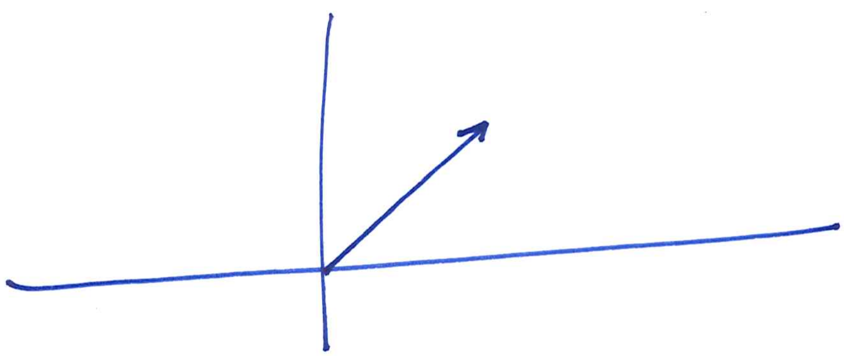
$\|\vec{b} - \vec{a}\|$.

$\vec{b} - \vec{a} = (b_1, b_2) - (a_1, a_2)$

$= (b_1 - a_1, b_2 - a_2)$

$\|\vec{b} - \vec{a}\| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$

the distance between vector $\vec{a}$ and $\vec{b}$.



Example

$\vec{a} = (4, 3)$ and $\vec{b} = (1, 2)$

$\|\vec{a} - \vec{b}\|$

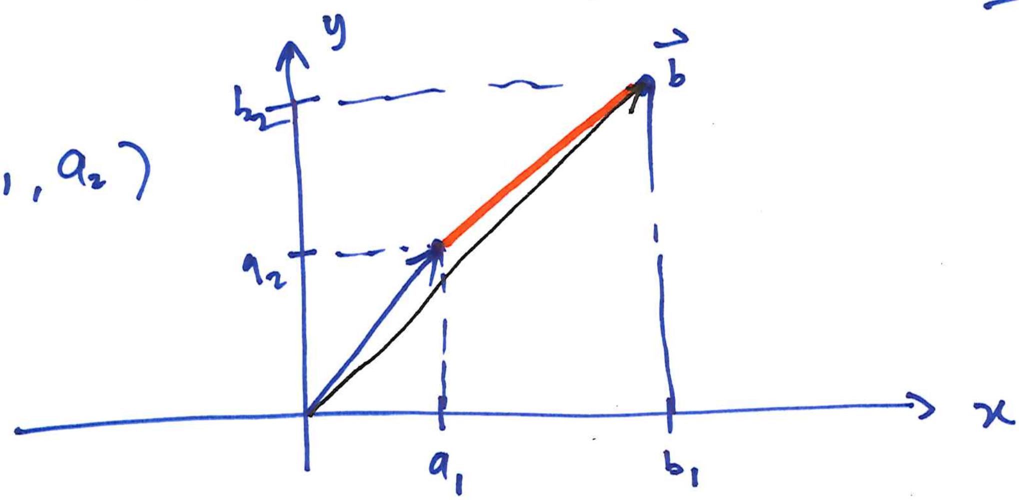
$\vec{a} - \vec{b} = (4, 3) - (1, 2) = (3, 1)$

$\|\vec{a} - \vec{b}\| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \underline{\underline{\sqrt{10}}}$

$\vec{b} - \vec{a} = (1, 2) - (4, 3) = \underline{(-3, -1)} = -1(3, 1)$

$\|\vec{b} - \vec{a}\| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9 + 1} = \underline{\underline{\sqrt{10}}}$

$\vec{a} = (a_1, a_2)$



The dot product

If vectors $\vec{a}$ and $\vec{b}$ in 2 D

$\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

The dot product of $\vec{a}$ and $\vec{b}$ is

$\vec{a} \cdot \vec{b} = (a_1, a_2) \cdot (b_1, b_2)$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Suppose $\vec{a} = (a_1, a_2, \ldots, a_n)$

$\vec{b} = (b_1, b_2, \ldots, b_n)$

Then

$\vec{a} \cdot \vec{b} = (a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n)$

$= a_1 b_1 + a_2 b_2 + \ldots + a_n b_n .$

properties of dot product

let $\vec{a}$ and $\vec{b}$ be vectors, and $\vec{0}$ be the zero vector.

(i) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(iv) $\alpha(\vec{a} \cdot \vec{b}) = (\alpha \vec{a}) \cdot \vec{b}$ $\qquad \alpha$ — scalar

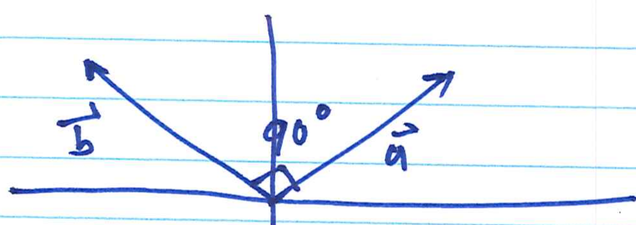
(v) $\vec{0} \cdot \vec{a} = 0$

(vi) $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$

θ is the angle between $\vec{a}$ and $\vec{b}$

(vii) $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are orthogonal

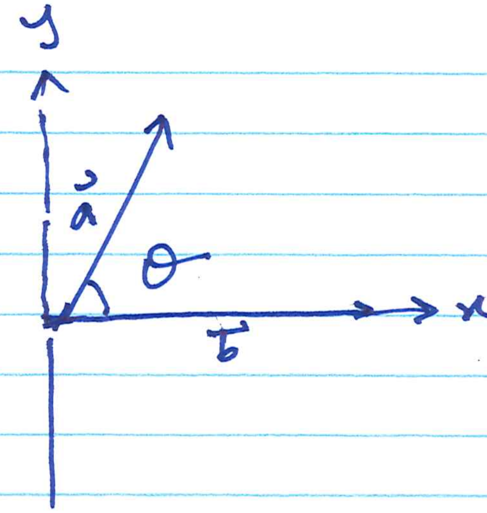
Def. Two vectors are orthogonal if they are perpendicular.



$\vec{a}$ and $\vec{b}$ are orthogonal.

property 6

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$ ✓



Suppose $\theta = 90^\circ$ or $\pi/2$ radian

$\cos(90) = 0$

$\vec{a}\cdot\vec{b} = 0$

Examples

$\vec{a} = (4, 3)$ and $\vec{b} = (1, 2)$

find the angle between $\vec{a}$ and $\vec{b}$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$\vec{a}\cdot\vec{b} = (4,3)\cdot(1,2) = 4+6 = 10$

$\|\vec{a}\| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

$\|\vec{b}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \dfrac{10}{5\sqrt{5}} = \dfrac{2}{\sqrt{5}}$

$\theta = 26.57^\circ$