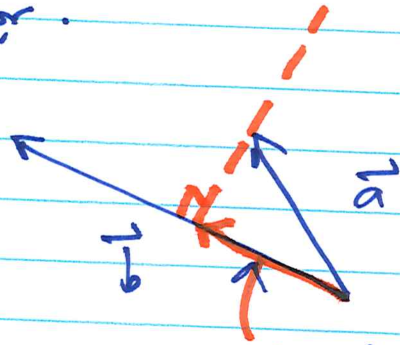
Last day

* projection of a vector in the direction of another vector.



$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

Example!

Let $\vec{a} = (1, 2, 4)$, $\vec{b} = (2, 1, 1)$. Find the projection of $\vec{a}$ in the direction of $\vec{b}$.

Solution

$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

$$\vec{a}\cdot\vec{b} = (1, 2, 4)\cdot(2, 1, 1) = 2 + 2 + 4 = 8$$

$$\|\vec{b}\| = \sqrt{2^2+1^2+1^2} = \sqrt{4+1+1} = \sqrt{6}, \quad \|\vec{b}\|^2 = 6$$

$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{8}{6}\right)(2, 1, 1) = \frac{4}{3}(2, 1, 1) = \left(8/3, 4/3, 4/3\right)$$

* Let $\vec{a}$ be vector, a unit vector in the direction $\vec{a}$ is given by

$$\underline{\underline{\hat{a}}} = \frac{\vec{a}}{\|\vec{a}\|}$$

* determinant of matrices

In 2D if $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$

$\det(A) = a_1 b_2 - a_2 b_1$

In 3D

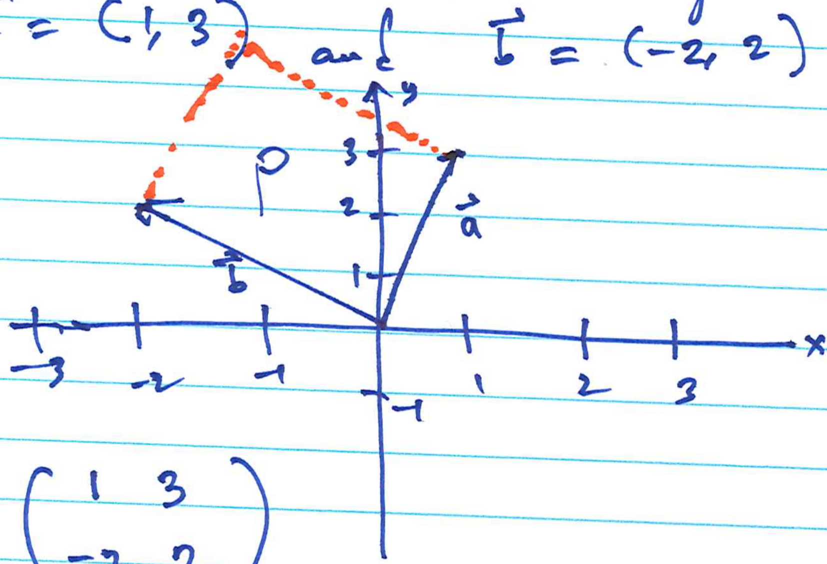
$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\det(A) = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - c_1 b_3) + a_3 (b_1 c_2 - b_2 c_1)$$

$\underline{\underline{\text{NB}}}$ Determinant can be computed for square matrices only.

Example:

Find the area of the parallelogram whose sides are $\vec{a} = (1, 3)$ and $\vec{b} = (-2, 2)$



Let $A = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$

$\text{Area}(P) = |\det(A)|$

$\det(A) = 1(2) - 3(-2) = 2 + 6 = 8$

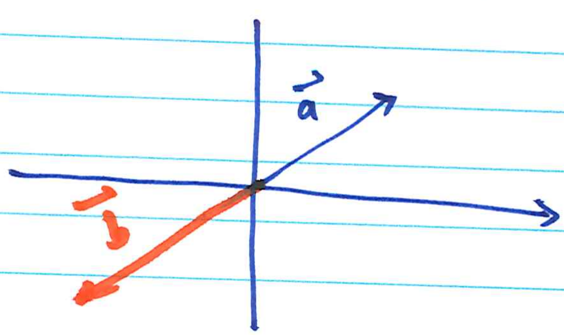
$\text{Area}(P) = |8| = \underline{\underline{8}}$

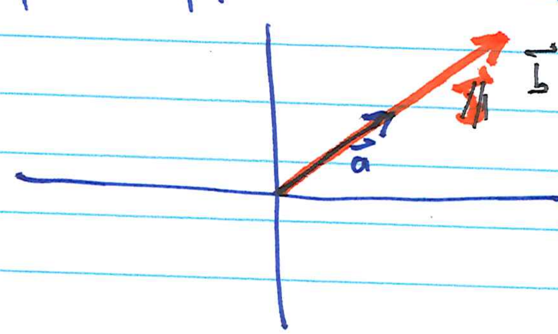
properties of determinant

Let $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$

Let $\vec{a} = (a_1, a_2)$ $\quad \vec{b} = (b_1, b_2)$

① $\det(A) = 0$ if the vectors $\vec{a}$ and $\vec{b}$ are parallel, ~~layer~~ lying on each other, or ~~are in~~ ~~are opposite of each~~ other are pointing in opposite direction





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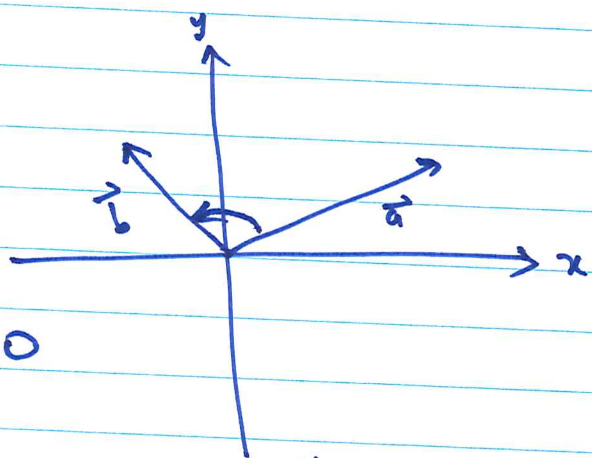
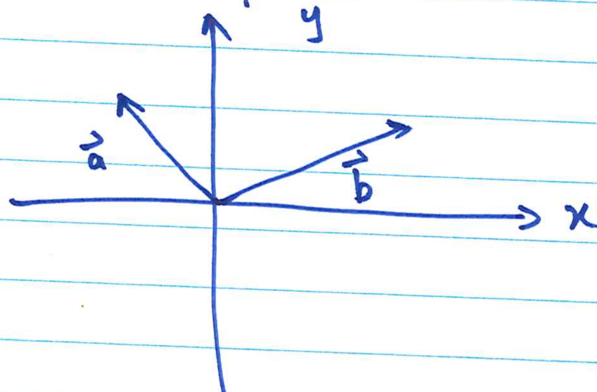


$$A = \begin{pmatrix} — \vec{a} — \\ — \vec{b} — \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b^2 \end{pmatrix}$$

$\det(A) > 0$

*



$\det(A) < 0$

<u>Example:</u>

Let $\vec{a} = (-2, 2)$ and $\vec{b} = (1, 3)$

$$A = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}$$

<u>Exercise</u>: $\vec{a} = (-1, 2)$ and $\vec{b} = (3, 2)$

Find the area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

<u>Solution</u>

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\det(A) = (-1)(2) - (2)(3) = -2 - 6 = -8$$

$$\text{Area} = |\det(A)| = |-8| = 8$$

The cross product

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

The cross product of $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \hat{i} (a_2 b_3 - a_3 b_2) - \hat{j} (a_1 b_3 - a_3 b_1) + \hat{k} (a_1 b_2 - a_2 b_1)$$

$$= \Big((a_2 b_3 - a_3 b_2), (a_3 b_1 - a_1 b_3), (a_1 b_2 - a_2 b_1) \Big)$$

Note

* ~~cros~~ the cross product of two vector is a vector
* Cross product can only be computed for vectors in 3D.

Example: Let $\vec{a} = (1, 3, 4)$ and $\vec{b} = (2, 4, 1)$

Find $\vec{a} \times \vec{b}$

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{vmatrix}$$

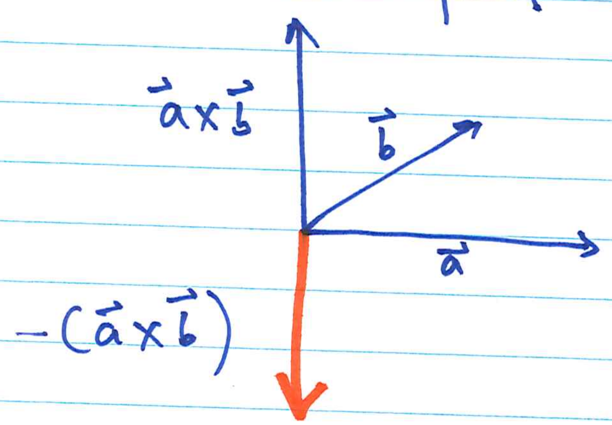
$$= \hat{i}\,(3-16) - \hat{j}\,(1-8) + \hat{k}\,(4-6)$$

$$\vec{a} \times \vec{b} = -13\hat{i} + 7\hat{j} - 2\hat{k} = (-13, 7, -2).$$

Properties of Cross product

Let $\vec{a}$ and $\vec{b}$ be vector in 3D.

1. $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$



Question:

what is $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

② $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$

Also, $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$

③ The vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ obey the right hand rule.

Additional properties of cross product

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in 3D.

(i) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$

(iii) $\alpha(\vec{a} \times \vec{b}) = (\alpha\vec{a}) \times \vec{b} = \vec{a} \times (\alpha\vec{b})$

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(v) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Example: Let $\vec{a} = (2, 3, -1)$ and $\vec{b} = (1, -1, 4)$

(a) Find the area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

Solution

$$\text{Area} = \|\vec{a} \times \vec{b}\|$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & 4 \end{vmatrix} = \hat{i}(12-1) - \hat{j}(8+1) + \hat{k}(-2-3)$$

$$= 11\hat{i} - 9\hat{j} - 5\hat{k}$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{11^2 + (-9)^2 + (-5)^2} = \sqrt{227}$$

$$\text{Area} = \sqrt{227}$$

(b) Find the angle between $\vec{a}$ and $\vec{b}$

we have

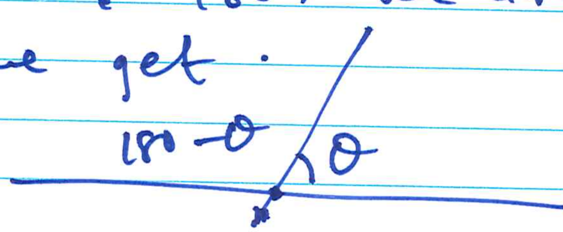
$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \quad (1) \qquad , \qquad \sin\theta = \frac{\|\vec{a} \times \vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|} \quad (2)$$

θ is the angle between $\vec{a}$ and $\vec{b}$

~~Here the~~ we shall use ~~Area~~ (1) because

$$\sin(\theta) = \sin(180 - \theta)$$

and when we take the arcsin of a value where the result is between 0 and 180, we are not sure of the angle we get.



Let us prove property 5

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Let $\vec{a}=(a_1, a_2, a_3)$, $\vec{b}=(b_1, b_2, b_3)$, and $\vec{c}=(c_1, c_2, c_3)$

L.H.S. $\vec{a}\cdot(\vec{b}\times\vec{c})$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - c_1b_3) + \hat{k}(b_1c_2 - c_1b_2)$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1, a_2, a_3)\cdot\left((b_2c_3 - b_3c_2), (c_1b_3 - b_1c_3), (b_1c_2 - c_1b_2)\right)$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(c_1b_3 - b_1c_3) + a_3(b_1c_2 - c_1b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2c_1b_3 - a_2b_1c_3 + a_3b_1c_2 - a_3c_1b_2 \quad (1)$$

R.H.S

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) \; \hat{k}(a_1b_2 - a_2b_1)$$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1) \quad (2)$$

$$= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$$

Since (1) = (2), $\therefore$ $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

Continuation of example

$\vec{a} = (2, 3, -1) \quad \vec{b} = (1, -1, 4)$

we want to compute the angle between these vectors.

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

$\vec{a} \cdot \vec{b} = (2, 3, -1) \cdot (1, -1, 4) = 2 + (-3) + (-4) = 2 - 7 = -5$

$\|\vec{a}\| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

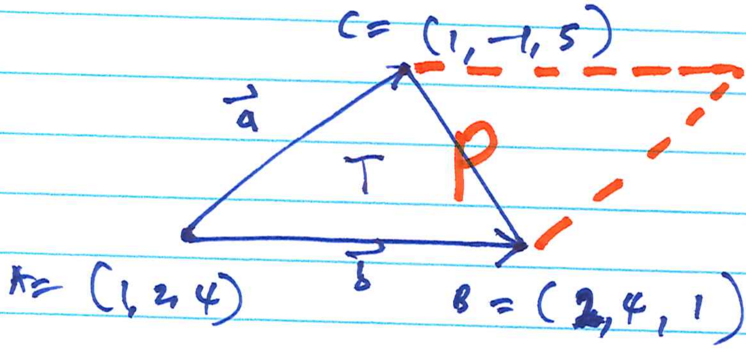
$\|\vec{b}\| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{1 + 1 + 16} = \sqrt{18}$

$$\cos(\theta) = \frac{-5}{(\sqrt{14})(\sqrt{18})} = \frac{-5}{\sqrt{252}} = -0.31497$$

$$\theta = \arccos(-0.31497) = 108.359^\circ$$

Example: Find the area of a triangle whose vertizes are $(1,2,4)$, $(1,-1,5)$ and $(2,4,1)$

Solution:



$$\boxed{\text{Area}(T) = \frac{\text{Area}(P)}{2}}$$

$$\vec{a} = (1,-1,5) - (1,2,4) = (0,-3,1)$$

$$\vec{b} = (2,4,1) - (1,2,4) = (1,2,-3)$$

$$\text{Area}(P) = \|\vec{a} \times \vec{b}\|$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \hat{i}(9-2) - \hat{j}(0-1) + \hat{k}(0-(-3))$$

$$\vec{a} \times \vec{b} = (7,1,3)$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{7^2+1^2+3^2} = \sqrt{49+1+9} = \sqrt{59}$$

$$\text{Area}(P) = \|\vec{a} \times \vec{b}\| = \sqrt{59}$$

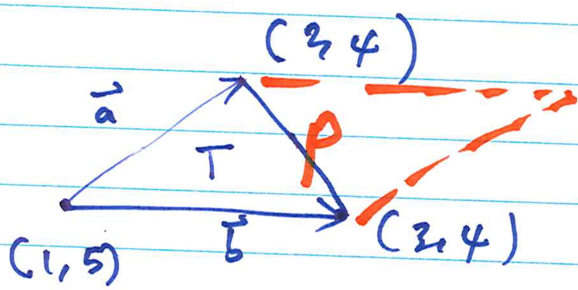
$$\text{Area}(T) = \frac{\text{Area}(P)}{2} = \frac{\sqrt{59}}{2} = \underline{\underline{3.8406}}$$

Example:

Find the area of a triangle whose vertices are
~~(1,5), (3,4), and (3,4)~~.

(1, 5), (3, 4), and (2, 4)

Solution:



$$\vec{a} = (2,4) - (1,5) = (1,-1)$$

$$\vec{b} = (3,4) - (1,5) = (2,-1)$$

$$\text{Area}(T) = \frac{\text{Area}(P)}{2} \qquad B = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\text{Area}(P) = |\det(A)| \quad \text{where } A = \begin{pmatrix} 1 & -1 \\ 2, & -1 \end{pmatrix}$$

$$\det(A) = 1(-1) - (-1)(2) = -1 + 2 = 1$$

$$\text{Area}(P) = |\det(A)| = \underline{\underline{1}}.$$

$$\text{Area}(T) = \frac{\text{Area}(P)}{2} = \underline{\underline{\frac{1}{2}}}$$

The Triple product

$\vec{a} = (a_1, a_2, a_3)$
$\vec{b} = (b_1, b_2, b_3)$
$\vec{c} = (c_1, c_2, c_3)$

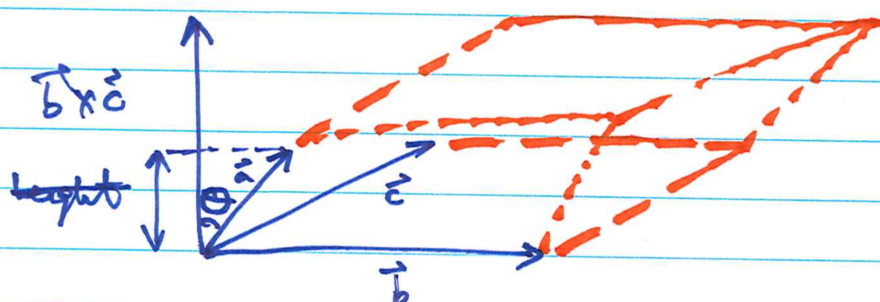
Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in 3D. The triple product of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by

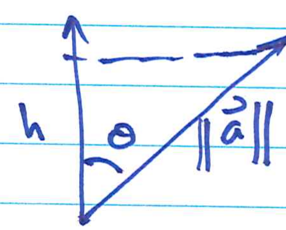
$$\underline{\underline{\vec{a}\cdot(\vec{b}\times\vec{c})}} = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

spanned

Consider the parallelepiped spanned by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$

height ~~height~~



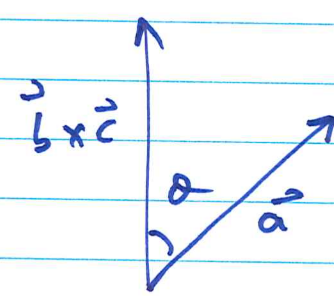


Volume of the parallelepiped = base area × height

$\cos\theta = \dfrac{h}{\|\vec{a}\|} \Rightarrow h = \|\vec{a}\|\cos\theta$

base area $= \|\vec{b}\times\vec{c}\|$

volume $= \|\vec{a}\|\cos\theta\,\|\vec{b}\times\vec{c}\|$ — 1



$\cos(\theta) = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\|\vec{a}\|\,\|\vec{b}\times\vec{c}\|} \Rightarrow |\vec{a}\cdot(\vec{b}\times\vec{c})| = \|\vec{a}\|\cos\theta\,\|\vec{b}\times\vec{c}\|$ 2

from 1 and 2, volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

We say vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfy the right hand rule if the triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c})$$

is positive.

<u>Example:</u> Find the volume of the parallelepiped spanned by vectors $\vec{a}=(1,4,-1)$, $\vec{b}=(2,1,3)$ and $\vec{c}=(1,4,9)$.

<u>Solution:</u>

volume of parallelepiped $= \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 4 & -1 \\ 2 & 1 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 1(9-12) - 4(18-3) + (-1)(8-1)$$

$$= -3 - 4(15) - 7$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = -70$$

$\therefore$ volume $= \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| = |-70| = \underline{\underline{70}}$.

Que: does $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfy the right hand rule?

$\underline{\underline{\text{NO}}}$ because $\vec{a}\cdot(\vec{b}\times\vec{c}) < 0$

Example: Find the volume of the parallelepiped spanned by the vectors

$$\vec{a} = (3, 2, 1), \quad \vec{b} = (-1, 3, 0), \text{ and } \vec{c} = (2, 2, 5)$$

Solution

$$\text{volume} = \cancel{|\vec{a}\cdot(\vec{b}\times\vec{c})|} = \cancel{\det\begin{pmatrix} 3 & 2 & 1 \\ -1 & & \end{pmatrix}}$$

$$= |\vec{b}\cdot(\vec{a}\times\vec{c})|$$

$$\vec{b}\cdot(\vec{a}\times\vec{c}) = \begin{pmatrix} -1 & 3 & 0 \\ 3 & 2 & 1 \\ 2 & 2 & 5 \end{pmatrix} = -1(8) - 3(13) + 0$$

$$= -8 - 39 = -47$$

$$\therefore \text{volume} = |\vec{b}\cdot(\vec{a}\times\vec{c})| = |-47| = \underline{\underline{47}}.$$

LINES and planes

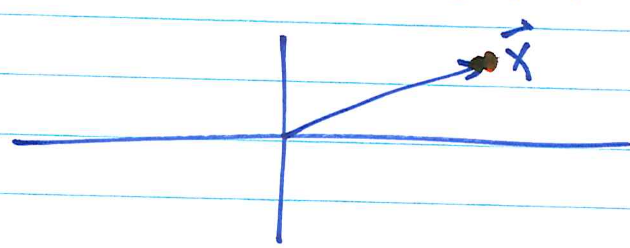
In this section, we will look at
- points
- lines
- planes
- space

in 2D and 3D.

Each of these can be written ~~either in~~ two forms:
- parametric form
  - the sets are written using vectors that are parallel to them
- equation form
  - ~~they~~ the sets are written using vectors that are orthogonal to the set.

~~Notation~~ Notation

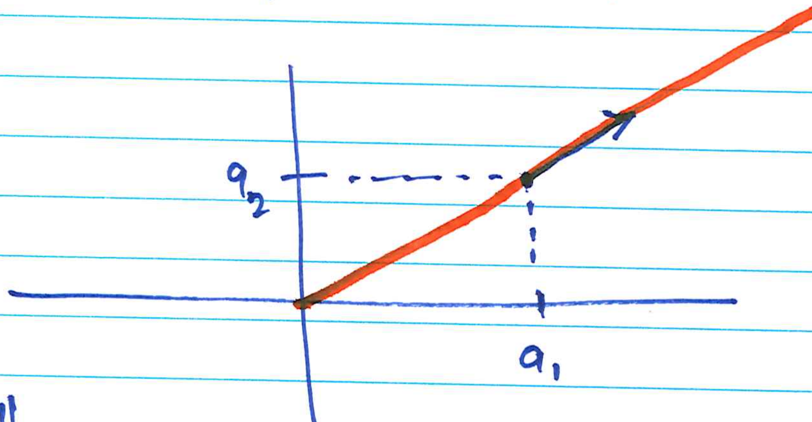
A point '$\vec{x}$' means the point at the head of a vector $\vec{x}$ whose tail is at the origin.



Lines in 2D

- parametric form

Let us consider a line passing through the origin. Let $\vec{a} = (a_1, a_2)$ be a vector in the direction of the line.



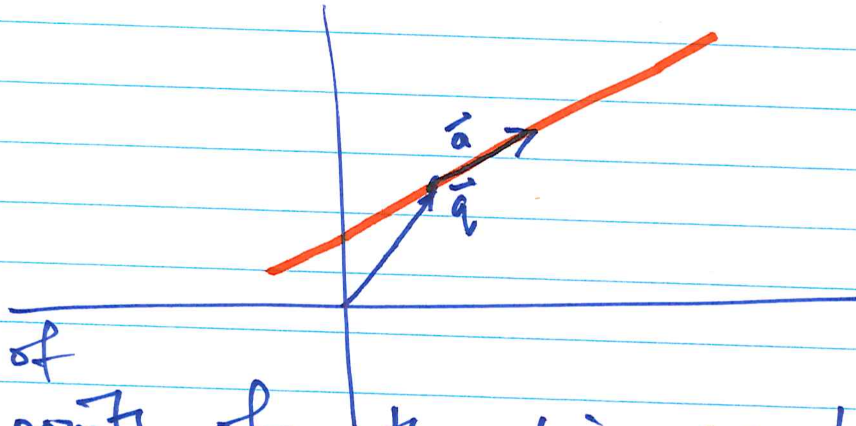
~~Any~~ ~~Any~~ All points on this line can be written in the form

$\vec{x} = t\,\vec{a}$ where t is some number.

The number t is called a parameter.

general case

Let $\vec{q}$ be a point on the line and $\vec{a}$ be the direction of the line



~~Each~~ Each of ~~all~~ the points of the line can be writte in parametric form as

$$\vec{X} = \vec{q} + t\,\vec{a} \quad \text{for some number } t.$$

Wenjia - Sally.