Last day

- Linear transformation

A transformation is linear if it satisfies

$$T(\alpha \vec{X} + \beta \vec{Y}) = \alpha T(\vec{X}) + \beta T(\vec{Y})$$

where $\vec{X}$ and $\vec{Y}$ are vectors and α, β are scalars.

- Rotation in 2D.

$$\text{Rot}_{\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for counter clockwise rotation.

- Projection in 2D

$$\text{Proj}_{\vec{a}} = \frac{1}{\|\vec{a}\|^2} \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix}$$

$$\text{Proj}_{\theta} = \frac{1}{2} \begin{pmatrix} 1+\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1-\cos(2\theta) \end{pmatrix}$$

Example: Find the vector obtained by projecting a vector $\vec{x} = (4, 1)$ in the direction of another vector $\vec{a}$ that makes an angle of $45°$ with the x-axis.

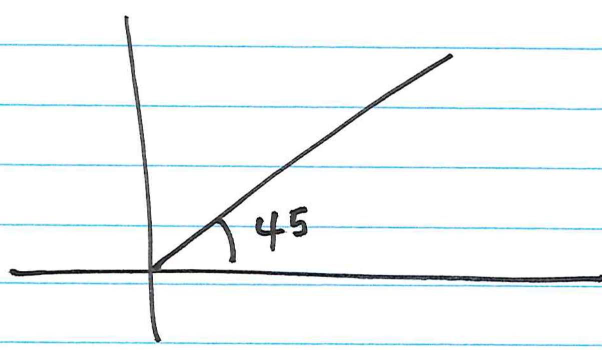
we have

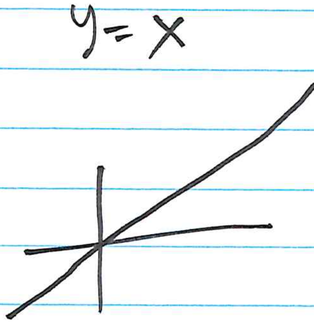
$$\text{Proj}_\theta = \frac{1}{2}\begin{pmatrix} 1+\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1-\cos(2\theta) \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \theta = 45, \quad \cos(90) = 0, \quad \sin(90) = 1$$

$$\text{Proj}_\theta \vec{x} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

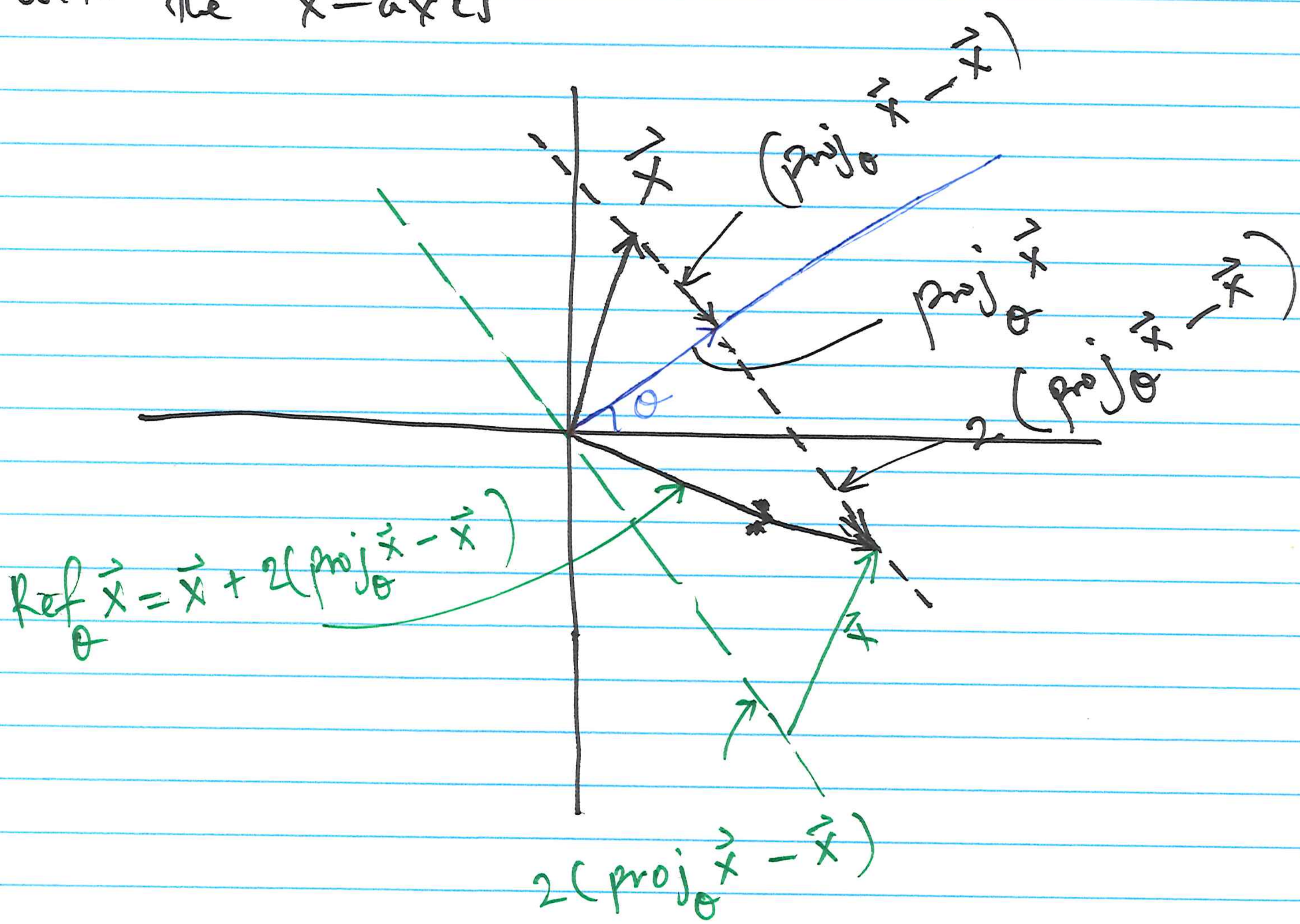
$$= \frac{1}{2}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}$$





REFLECTION IN 2D

Let $\vec{x}$ be a vector in 2D, Consider of projection of ~~vect~~ $\vec{x}$ in the direction of another vector that makes an angle of θ with the x-axis



$$\mathrm{Ref}_\theta \vec{x} = \vec{x} + 2(\mathrm{Proj}_\theta \vec{x} - \vec{x})$$

$$= 2\,\mathrm{Proj}_\theta \vec{x} - \vec{x}$$

$$= (2\,\mathrm{Proj}_\theta - I)\vec{x}$$

, I is an identity matrix

we know

$$\mathrm{Proj}_\theta = \frac{1}{2}\begin{pmatrix} 1+\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1-\cos(2\theta) \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Ref}_\theta = \begin{pmatrix} 1+\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1-\cos(2\theta) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Ref}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

This is the <u>reflection matrix</u>!

Example:

$$f(x) = 5x^3 + x^2 = (5x+1)\,x^2$$

Identity matrix

An identity matrix is a matrix with '1' on the diagonal and zero else where.

E.g In 3D

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5D

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In matlab

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eye(5)
speye(5)
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Example: Find the reflection of the vector $\vec{x} = (5, 1)$ across a line that makes an angle of 45° with the x-axis.

we know

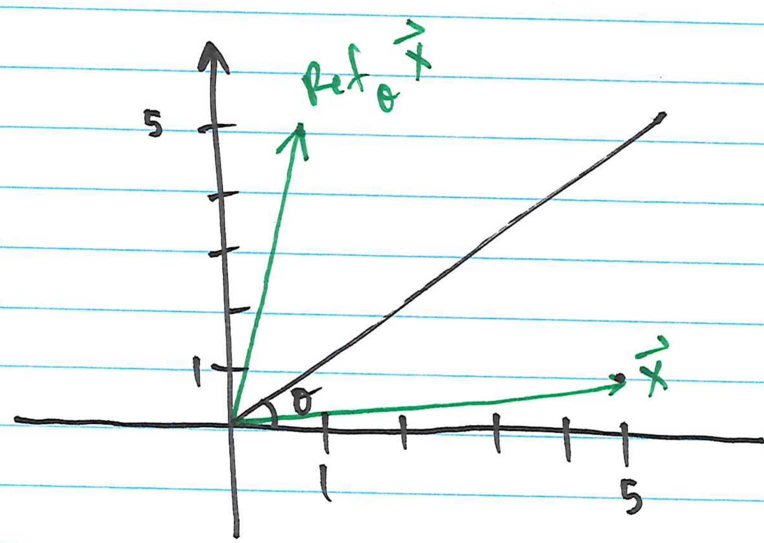
$$\text{Ref}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$\theta = 45^\circ$

$\cos(90) = 0, \quad \sin(90) = 1$

$$\therefore \text{Ref}_\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Ref}_\theta \vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS

Let T be a linear ~~linea~~ transformation and ~~names~~

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be the standard basis vectors in $\mathbb{R}^3$.

Let $\vec{x} \in \mathbb{R}^3$, then $\vec{x}$ can be written as a linear combination of e_1, e_2, and e_3.

Eg

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

Example: Find the matrix representation of the linear transformation $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} 2x_1 + 7x_2 + 5x_3 \\ 3x_1 + 4x_2 + x_4 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_4 \end{pmatrix}$$

We want constrict the matrix

$$T = \begin{pmatrix} | & | & | & | \\ T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ | & | & | & | \end{pmatrix}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$T(e_1) = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 7 \\ 4 \\ 1 \\ 0 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 5 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ $T(e_4) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$

$$T = \begin{pmatrix} 2 & 7 & 5 & 0 \\ 3 & 4 & 0 & 1 \\ 1 & 1 & 3 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}$$

Letus apply the transformation on $\vec{X}$,

$$T(\vec{x}) = T(x_1 e_1 + x_2 e_2 + x_3 e_3)$$

$$= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) \quad \text{——} \quad (1)$$

Let

$$T = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix}$$ ✓

the product $T\vec{X} = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$T\vec{X} = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) \quad \text{——} \quad (2)$$

Observe that (1) = (2).

Therefore the ~~transfor~~ matrix of the transformation

is $T = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix}$

Composition of Linear transformations

Recall that if we have functions $f(x)$ and $g(x)$

$$f(g(x))$$

is the composition of the two function.

let us extend the idea to transformations.

Let T and S be linear transformations, then

$$S(T(\vec{X}))$$

is a composition of the two transformations.

The composition of two linear transformations is also a linear transformation.

Let $\vec{x}$ and $\vec{y}$ be vectors and α, β be scalars.

$$T(\alpha \vec{x} + \beta \vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$$

$$S\left(T(\alpha \vec{x} + \beta \vec{y})\right) = S\left(\alpha T(\vec{x}) + \beta T(\vec{y})\right)$$

$$= \alpha S\left(T(\vec{x})\right) + \beta S\left(T(\vec{y})\right)$$

Clearly, this transformation is linear.

Matrix of composition of linear transformations.

Let S and T be linear transformations

Let $\hat{T}$ be the matrix for T

$\hat{S}$ be the matrix for S

Let $\vec{x}$ be vector,

$T(\vec{x})$ is equivalent $\hat{T}\vec{x}$

$S(\vec{x})$ " " $\hat{S}\vec{x}$

$S(T(\vec{x}))$ is equivalent to $\hat{S}(\hat{T}\vec{x})$

~~Associte~~ Associative property

$$A(BC) = (AB)C$$

$$\Rightarrow \quad \hat{S}(\hat{T}\vec{x}) = (\hat{S}\hat{T})\vec{x}$$

$\Rightarrow$ $A = \hat{S}\hat{T}$ is the matrix of the composition.

Example: Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T((x_1, x_2)) = \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + x_2 \end{pmatrix}$$

and $S: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$S((x_1, x_2)) = \begin{pmatrix} x_1 + 4x_2 \\ 3x_1 + x_2 \end{pmatrix}$$

Ques: Find the matrix for the composition $S(T(\vec{x}))$

$$\hat{T} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}$$

$$A = \hat{S}\hat{T} = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 7 & 10 \end{pmatrix}$$

Check Let $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$T(\vec{x}) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \quad S(T(\vec{x})) = \begin{pmatrix} 19 \\ 24 \end{pmatrix} \checkmark$$

$$(\hat{S}\hat{T})\vec{x} = \begin{pmatrix} 6 & 7 \\ 7 & 10 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 12+7 \\ 14+10 \end{pmatrix} = \begin{pmatrix} 19 \\ 24 \end{pmatrix} \checkmark$$

Check if $S(T(\vec{x})) = T(S(\vec{x}))$?

$$\hat{T}\hat{S} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 4 & 5 \end{pmatrix}$$

$$(\hat{T}\hat{S})\vec{x} = \begin{pmatrix} 11 & 11 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 33 \\ 13 \end{pmatrix}$$

$S(T(\vec{x})) \neq T(S(\vec{x}))$. This can also be seen by taking note that $\hat{S}\hat{T} \neq \hat{T}\hat{S}$