Last day

* lines in 2D

* parametric form

a line passing through the point $\vec{q}$ in the direction of vector $\vec{a}$ is given by

$$\vec{X} = \vec{q} + t\vec{a} \quad \cancel{\text{for}} \text{ some } t \in \mathbb{R}.$$

* equation form

$$\vec{X}\cdot\vec{b} = \vec{q}\cdot\vec{b}$$

where $\vec{b}$ is the vector in the direction orthogonal to the line

* lines in 3D

* parametric form

a line that passes through point $\vec{q}$ in the direction of $\vec{a}$ is given by

$$\vec{X} = \vec{q} + t\vec{a}$$

* equation form of the same line is given by

$$\left.\begin{aligned}(\vec{x}-\vec{q})\cdot\vec{b}_1 &= 0\\ (\vec{x}-\vec{q})\cdot\vec{b}_2 &= 0\end{aligned}\right\} \text{---} (+)$$

where the vectors $\vec{b}_1$ and $\vec{b}_2$ are on the plane orthogonal to the line

$$\left.\begin{aligned}\vec{x}\cdot\vec{b}_1 &= \vec{q}\cdot\vec{b}_1\\ \vec{x}\cdot\vec{b}_2 &= \vec{q}\cdot\vec{b}_2\end{aligned}\right\} \text{---} (*)$$

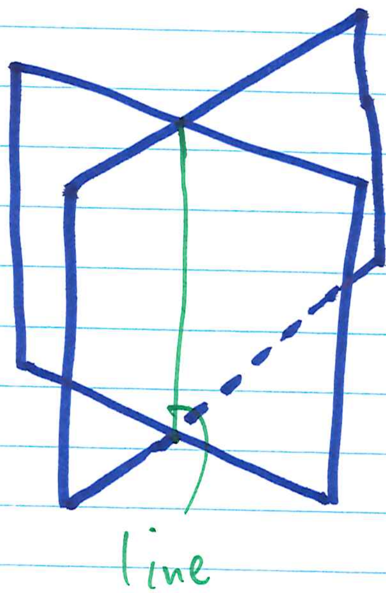
if $\vec{x} = (x_1, x_2, x_3)$, $\vec{b}_1 = (b_{11}, b_{12}, b_{13})$
$\vec{b}_2 = (b_{21}, b_{22}, b_{23})$,

$$\vec{q} = (q_1, q_2, q_3)$$

then $\overset{(+)}{\cancel{(**)}}$ becomes

$$(x_1 - a_1) b_{11} + (x_2 - a_2) b_{12} + (x_3 - a_3) b_{13} = 0$$

$$(x_1 - a_1) b_{21} + (x_2 - a_2) b_{22} + (x_3 - a_3) b_{23} = 0$$



<u>Note</u>

vectors $\vec{b}_1$ and $\vec{b}_2$ are ~~non~~

non-collinear

Example:

Find a parametric and an equation form of the line that passes through the points $(1,2,5)$ and $(2,3,4)$.

Solution

• Parametric

Let $A=(1,2,5)$

$B=(2,3,4)$

$$\vec{X}=\vec{q}+t\vec{a}$$

$$\vec{a}=B-A=(2,3,4)-(1,2,5)$$
$$=(1,1,-1)$$

and $\vec{q}=(1,2,5)$

$$\therefore \vec{X}=(1,2,5)+t(1,1,-1)$$

if $\vec{q}=(2,3,4)$, then

$$\vec{X}=(2,3,4)+t(1,1,-1),\ t\in\mathbb{R}.$$

* equation form

$$\vec{X} \cdot \vec{b}_1 = \vec{q} \cdot \vec{b}_1$$

$$\vec{X} \cdot \vec{b}_2 = \vec{q} \cdot \vec{b}_2$$

We know $\vec{q}$, let $\vec{q} = (1, 2, 5)$

We know $\vec{a}$,

Let ~~$\mathbb{R}$~~ (x, y) be orthogonal to $\vec{a}$, then

$\cancel{(x, y) \cdot \vec{a} = 0}$ $\qquad (x, y, z) \cdot \vec{a} = 0$

$\cancel{(x, y) \cdot (a_1, a_2) = 0}$ $\qquad \vec{a} = (a_1, a_2, a_3)$

$\cancel{a_1 x \pm a_2 y}$ $\qquad (x, y, z) \cdot (a_1, a_2, a_3) = 0$

$$a_1 x + a_2 y + a_3 z = 0$$

$\vec{a} = (1, 1, -1)$

$$x + y - z = 0$$

guess x and y and solve for z.

let $x = 1$, $y = 2$, $1 + 2 - z = 0$

$z = 3$

$\vec{b}_1 = (1, 2, 3)$

let $x = 3$, $y = 1$, $3 + 1 - z = 0 \Rightarrow z = 4$

$\vec{b}_2 = (3, 1, 4)$.

We have $\vec{x} = (x_1, x_2, x_3)$

$$\vec{x}\cdot\vec{b}_1 = \vec{q}\cdot\vec{b}_1$$

$$\Rightarrow (x_1, x_2, x_3)\cdot(1, 2, 3) = (1, 2, 5)\cdot(1, 2, 3)$$

$$x_1 + 2x_2 + 3x_2 = 20$$

$$\vec{x}\cdot\vec{b}_2 = \vec{q}\cdot\vec{b}_2$$

$$\Rightarrow (x_1, x_2, x_3)\cdot(3, 1, 4) = (1, 2, 5)\cdot(3, 1, 4)$$

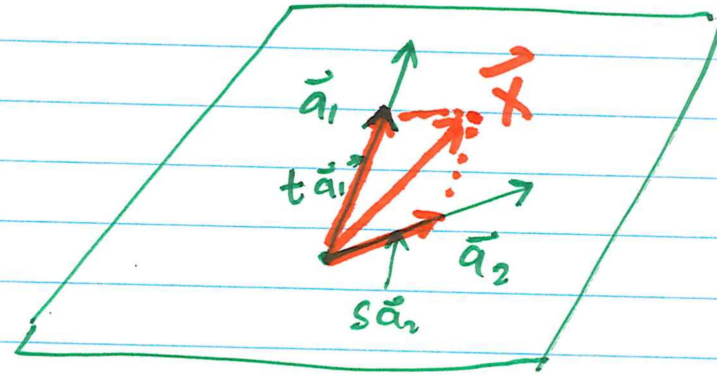
$$3x_1 + x_2 + 4x_3 = 25$$

∴ an ~~the~~ equation form of the line is

$$x_1 + 2x_2 + 3x_3 = 20$$

$$3x_1 + x_2 + 4x_3 = 25$$

plane S



* parametric form

$\vec{a}_1$ and $\vec{a}_2$ are noncollinear vectors on the plane

$$\vec{x} = t\vec{a}_1 + s\vec{a}_2$$

The parametric form of a plane that passes through the origin.

* suppose the plane passes through the point $\vec{q}$,

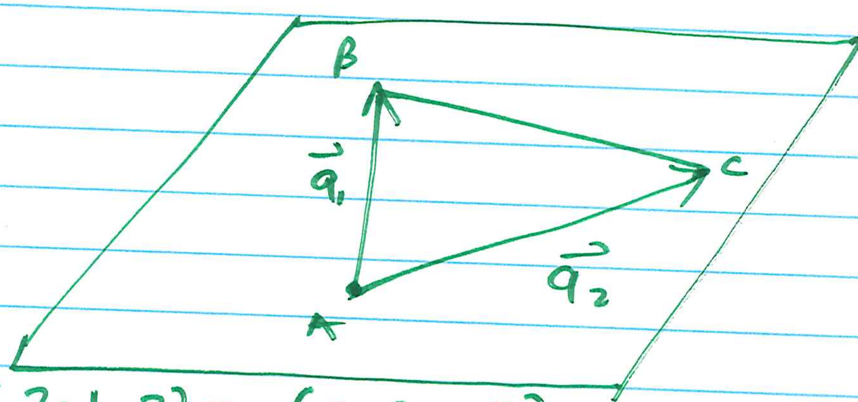
$$\vec{x} = \vec{q} + t\vec{a}_1 + s\vec{a}_2 \quad \text{for some } t, s$$

<u>Example:</u> find ~~the~~ a parametric form of the plane that passes through the points $(2,1,3)$, $(\cancel{4}\ 2,1,0)$ and $(2,3,4)$

$A = (2,1,3)$

$B = (2,1,0)$

$C = (2,3,4)$



$\vec{a}_1 = B - A = (2,1,0) - (2,1,3) = (0,0,-3)$

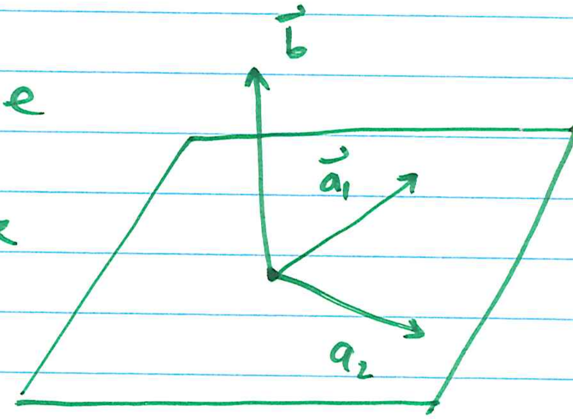
$\vec{a}_2 = C - A = (2,3,4) - (2,1,3) = (0,2,1)$

let $\vec{q} = (2,1,3)$

$\therefore \vec{x} = (2,1,3) + t(0,0,-3) + s(0,2,1)$

t and s are real numbers.

* equation form of a plane



Let $\vec{x}$ be a point on the plane

then

$$\vec{x} \cdot \vec{b} = 0$$

If $\vec{x} = (x_1, x_2, x_3)$, $\vec{b} = (b_1, b_2, b_3)$

Then $\vec{x} \cdot \vec{b} = 0$

$$\Rightarrow \quad x_1 b_1 + x_2 b_2 + x_3 b_3 = 0$$

Equation of a plane that passes through the origin.

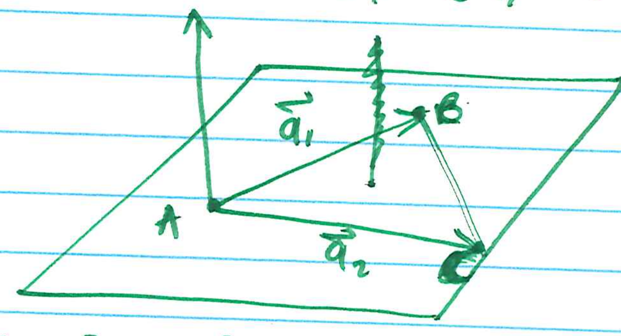
Suppose the plane passes through point $\vec{q}$, and $\vec{b}$ is orthogonal to the plane. If $\vec{x}$ is on the plane, then

$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

$$\vec{x} \cdot \vec{b} = \vec{q} \cdot \vec{b}$$

$$b_1 (x_1 - q_1) + b_2 (x_2 - q_2) + b_3 (x_3 - q_3) = 0$$

Example: Find the equation form of the plane that passes through the points $(2,3,1)$, $(1,-2,-1)$ and $(-1,2,3)$.



$A=(2,3,1)$, $B=(1,-2,-1)$

$C=(-1,2,3)$

$\vec{a}_1 = (1,-2,-1)-(2,3,1) = (-1,-5,-2)$

$\vec{a}_2 = (-3,-1,2)$

$$\vec{b} = \vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -5 & -2 \\ -3 & -1 & 2 \end{vmatrix} = \big((-10-2), -(-2-6), (1-15)\big)$$

$\vec{b} = (-12, 8, -14)$

let $\vec{x} = (x_1, x_2, x_3)$, $\vec{q} = (2,3,1)$

$$\vec{x}\cdot\vec{b} = \vec{q}\cdot\vec{b}$$

$$\Rightarrow (x_1,x_2,x_3)\cdot(-12,8,-14) = (2,3,1)\cdot(-12,8,-14)$$

$$\Rightarrow -12x_1 + 8x_2 - 14x_3 = -14$$

a parametric form of the plane is

$$\vec{x} = (2,3,1) + t(-1,-5,-2) + s(-3,-1,2).$$

Note

* the vector $\vec{b}$ is the normal vector to the plane
  if $\|\vec{b}\| = 1$, then $\vec{b}$ is called the unit normal vector

$$\hat{b} = \frac{\vec{b}}{\|\vec{b}\|}$$

Example: consider $x - y + 2z = 7$
① find the normal direction to the plane.

$$\vec{X} \cdot \vec{b} = \underbrace{\vec{q} \cdot \vec{b}}_{7} \quad , \quad \vec{b} - \text{normal vector}$$

$x - y + 2z = \vec{X} \cdot \vec{b}$ —— ✱1

$\vec{X} = (x, y, z)$, $\vec{b} = (b_1, b_2, b_3)$

$\vec{X} \cdot \vec{b} = (x, y, z) \cdot (b_1, b_2, b_3) = x b_1 + y b_2 + z b_3$ —— (✱2)

Compare ✱1 and ✱2

$b_1 = 1, \quad b_2 = -1, \quad b_3 = 2$

$\vec{b} = (1, -1, 2)$.

b (11) find a point on the plane.

Let $x = 5$, $y = 1$

$x - y + 2z = 7$

$5 - 1 + 2z = 7$

$4 + 2z = 7$

$2z = 3$

$z = 3/2$

$\therefore$ $(5, 1, 3/2)$ is on the plane.

Example:

Find a parametric form of the line of intersection of the planes

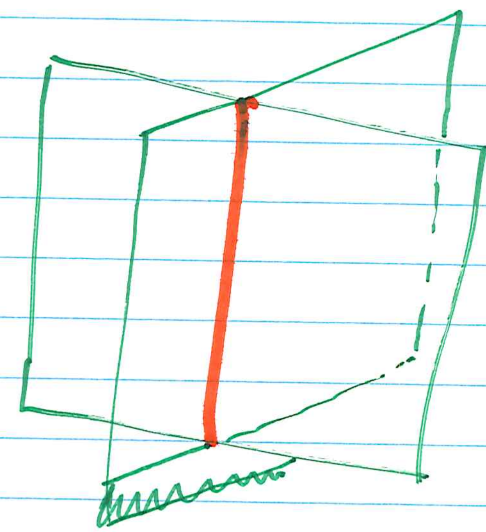
(1) $x + y + z = 2$ and $x - y + 2z = 7$ (2)

from (1)

$$x = 2 - y - z \quad (3)$$

from (2)

$$x = 7 + y - 2z \quad (4)$$



equate (3) and (4)

$$2 - y - z = 7 + y - 2z$$

$$2z - z = 7 - 2 + y + y$$

$$z = 5 + 2y \quad (5)$$

~~Let $y = t$, then~~

put (5) in (3)

$$x = 2 - y - (5 + 2y) = 2 - y - 5 - 2y = -3 - 3y$$

let $y = t$, $x = -3 - 3t$, $z = 5 + 2t$

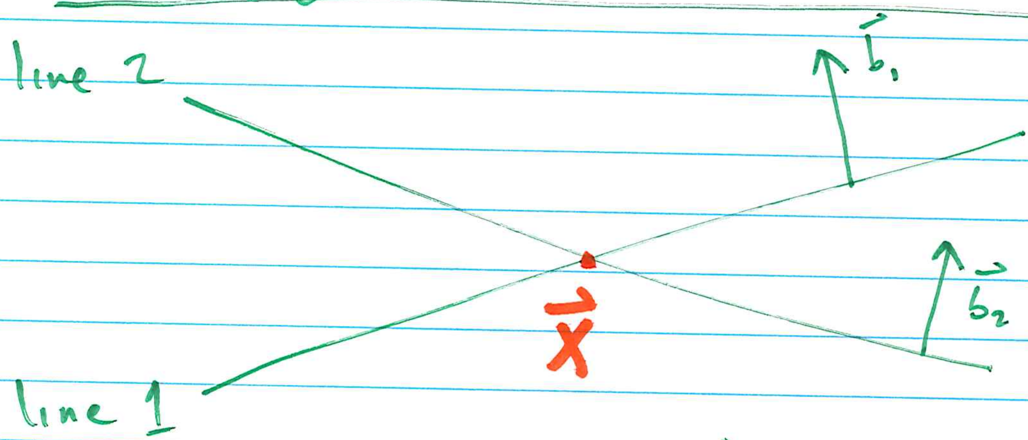
$$\vec{X} = (-3 - 3t, t, 5 + 2t)$$

$$= (-3, 0, 5) + (-3t, t, 2t)$$

$$\vec{X} = (-3, 0, 5) + t(-3, 1, 2), \quad t \in \mathbb{R}$$

LINEAR SYSTEMS

* Geometry of solutions to system of equations



$\vec{b}_1 = (b_{11}, b_{12})$ and $\vec{b}_2 = (b_{21}, b_{22})$ are non collinear vectors.

Let $\vec{x}$ be the point of intersection of line 1 and line 2.

$$\Rightarrow \quad \vec{x} \cdot \vec{b}_1 = 0$$
$$\vec{x} \cdot \vec{b}_2 = 0$$

Let $\vec{x} = (x_1, x_2)$

$$\left.\begin{array}{ll} \vec{x} \cdot \vec{b}_1 = 0 & \Rightarrow x_1 b_{11} + x_2 b_{12} = 0 \\ \vec{x} \cdot \vec{b}_2 = 0 & \Rightarrow x_1 b_{21} + x_2 b_{22} = 0 \end{array}\right\} \text{— } (1)$$

(1) is called a linear system of equations.

Therefore, $\vec{x}$ is a solution of system (1)

This implies that the solution of a 2 dimensional system of equation is the point of intersection of two lines.