Last day

* Complex eigenvalue and eigenvectors.

Let A is an $n\times n$ real matrix. If $\lambda \in \mathbb{C}$ an eigenvalue of A with eigenvector $\vec{X}$. Then the complex conjugate $\bar{\lambda}$ is also an eigenvalue of A and ~~the~~ ~~same~~ its corresponding eigenvector ~~the~~ is the complex conjugate of $\vec{X}$.

* How to compute ~~the~~ $\vec{X}_n$ for a random walk.

Example:

$$P = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{pmatrix}, \quad \vec{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{X}_n = P^n \vec{X}_0 = \vec{Y}_1 - \frac{1}{4}\left(\frac{1}{5}\right)^n \vec{Y}_2 + \left(\frac{1}{12}\right)\left(-\frac{1}{5}\right)^n \vec{Y}_3$$

where $\lambda = 1, \frac{1}{5}$, and $-\frac{1}{5}$ are the eigenvalues of P, with corresponding eigenvectors $\{\vec{Y}_1, \vec{Y}_2, \vec{Y}_3\}$, respectively.

We want to compute $\vec{X}_n$ for large n (i.e the state of the random walk after a long time)

Then we take the limit of $\vec{X}_n$ as $n \to \infty$.

i.e,

$$\lim_{n\to\infty} \vec{X}_n = \lim_{n\to\infty} \left[\vec{Y}_1 - \frac{1}{4}\left(\frac{1}{5}\right)^n \vec{Y}_2 + \left(\frac{1}{12}\right)\left(-\frac{1}{5}\right)^n \vec{Y}_3 \right]$$

$$= \vec{Y}_1$$

$\Rightarrow$ $\vec{Y}_1$ is the equilibrium probability of the random walk.

We observe that the probability is independent of the initial state of the network.

Theorem

Let P be a transition matrix of a random walk (i.e. the entries of P are non-negative and each column sums to 1). Then

(i) All the eigenvalues λ of P satisfy $|\lambda| \le 1$.

(ii) P has an eigenvalue $\lambda = 1$. If there is only one of such eigenvalues with $|\lambda| = 1$, then the corresponding eigenvector is a scalar multiple of the equilibrium ~~proper~~ probability.

(iii) All other eigenvalues of P satisfy $|\lambda| < 1$ and the entries of ~~those~~ their eigenvectors sum to zero. That is, the entries of each of the eigenvectors with eigenvalue $|\lambda| < 1$ sums to zero.

Example: Find the eigenvalues and eigenvectors of the transition matrix P of a random walk.

$$P = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$$

with $\vec{X}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Use the eigenanalysis to describe the long time behaviour of the random walk (or to find the equilibrium probability of the random walk).

Let λ be an eigenvalue of P, then

$$|P - \lambda I| = 0 .$$

$$\Rightarrow \left| \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1/4 - \lambda & 1/2 \\ 3/4 & 1/2 - \lambda \end{vmatrix} = 0$$

$$\left(1/4 - \lambda\right)\left(1/2 - \lambda\right) - 3/8 = 0$$

$$\frac{1}{8} - \frac{\lambda}{4} - \frac{\lambda}{2} + \lambda^2 - \frac{3}{8} = 0$$

$$\Downarrow$$

$$4\lambda^2 - 3\lambda - 1 = 0$$

$$(\lambda - 1)(4\lambda + 1) = 0$$

$$\Rightarrow \quad \lambda_1 = 1 \quad , \quad \lambda_2 = -\frac{1}{4}$$

$|\lambda_1| = 1$ and $|\lambda_2| = \frac{1}{4} < 1$ ✓.

Let us find the eigenvectors of P.

For $\lambda_1 = 1$, $(P - \lambda_1 I)\vec{V}_1 = \vec{0}$

$$\left(\begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3/4 & 1/2 \\ 3/4 & -1/2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -3/4 & 1/2 & 0 \\ 3/4 & -1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3/4 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]_{R_2 = R_2 + R_1}$$

$$-\tfrac{3}{4} \cdot v_1 + \tfrac{1}{2} v_2 = 0$$

$$-\tfrac{3}{4} v_1 = -\tfrac{1}{2} v_2$$

$$v_1 = \tfrac{2}{3} v_2$$

$$\therefore \vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \tfrac{2}{3} v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} \tfrac{2}{3} \\ 1 \end{pmatrix}, \quad v_2 \neq 0 \text{ is free.}$$

To get a probability vector, we take $v_2 = 3/5$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$$

For $\lambda_2 = -1/4$, $(P - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\Rightarrow \left(\begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 3/4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the reduced augmented matrix is $\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\frac{1}{2} v_1 + \frac{1}{2} v_2 = 0$

$\Rightarrow \quad v_1 = -v_2$

$$\vec{V}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad , v_2 \neq 0 \text{ is free.}$$

take $v_2 = 1$, $\vec{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Now we have the eigen analysis of P, i.e.

$$\lambda_1 = 1 \quad , \quad \vec{V}_1 = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$$

$$\lambda_2 = -1/4 \quad , \quad \vec{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The eigenvectors $\vec{V}_1$ and $\vec{V}_2$ are linearly independent,
$\Rightarrow$ that we can write $\vec{X}_0$ as a linear combination of $\vec{V}_1$ and $\vec{V}_2$.

$$\vec{X}_0 = \alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2$$

$$\alpha_1 \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by inspection $\alpha_1 = 1$, $\alpha_2 = -3/5$ works!

$$\vec{X}_0 = \vec{V}_1 + \left(-\tfrac{3}{5}\right)\vec{V}_2$$

$$\vec{X}_1 = P\vec{X}_0$$
$$= P\left(\vec{V}_1 + \left(-\tfrac{3}{5}\right)\vec{V}_2\right)$$
$$= P\vec{V}_1 + \left(-\tfrac{3}{5}\right)P\vec{V}_2$$

Since $\vec{V}_1$ and $\vec{V}_2$ are eigenvectors of P, then

$$P\vec{V}_1 = \lambda_1\vec{V}_1 \quad \text{and} \quad P\vec{V}_2 = \lambda_2\vec{V}_2$$

$$\Rightarrow \vec{X}_1 = \lambda_1\vec{V}_1 + \left(-\tfrac{3}{5}\right)\lambda_2\vec{V}_2$$

$$\vdots$$

$$\vec{X}_n = (\lambda_1)^n\vec{V}_1 + \left(-\tfrac{3}{5}\right)(\lambda_2)^n\vec{V}_2 = P^n\vec{X}_0$$

To get the long time behaviour of the network, we take the limit of $\vec{X}_n$ as $n \to \infty$.

i.e, $$\lim_{n\to\infty}\vec{X}_n = \lim_{n\to\infty}\left[(\lambda_1)^n\vec{V}_1 + \left(-\tfrac{3}{5}\right)\lambda_2^n\vec{V}_2\right]$$

$$\lim_{n \to \infty} \vec{x}_n = \lim_{n \to \infty} \left[(1)^n \vec{v}_1 + \left(-\tfrac{3}{5}\right)\left(-\tfrac{1}{4}\right)^n \vec{v}_2 \right]$$

$\left(-\frac{1}{4}\right)^n \to 0$ as $n \to \infty$.

$$\therefore \lim_{n \to \infty} \vec{x}_n = \vec{v}_1$$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$ is the equilibrium probability of the ~~network~~ random walk.

Example: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

Set $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(\lambda-3)(\lambda-1) = 0$$

the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

For the eigenvector of $\lambda_1 = 1$, $(A - \lambda_1 I)\vec{X} = \vec{0}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

this gives $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad X = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix}$

$\Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$ $\quad \vec{X} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

take $x_3 = 1$

$$\therefore \vec{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Similarly, for $\lambda_2 = 2$, $(A - \lambda_2 I)\vec{x} = \vec{0}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system, we get

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Let us construct a matrix whose columns are the eigenvectors of A.

$$T = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

We can write A as

~~$A = PDP^{-1}$~~ (diagonalization of A)

$$A = TDT^{-1}$$

where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

Then,

$$A^n = TD^nT^{-1}$$

Details

$$A = TDT^{-1}$$

$$A^2 = (TDT^{-1})(TDT^{-1}) = TD(T^{-1}T)DT^{-1}$$

$$= (TD)I(DT^{-1}) = T(DD)T^{-1} = TD^2T^{-1}$$

$\therefore A^2 = TD^2T^{-1}$ (we have use associative property for matrices)

using the same approach we can show that

$$A^n = TD^nT^{-1}$$