Last day

Consider a collection vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$, then the linear combination of the vectors is

$$LC = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_n \vec{a}_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars

- $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly dependent if $\exists$ $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$LC = 0$$

- $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly independent if

$$LC = 0$$

only when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Remarks

- In 2D, ~~we say~~ if two vectors are linearly dependent then they are collinear or parallel to each other.

- In 3D, they are ~~they~~ linearly dependent if the lie on the same plane.

Example: write $(5, 1)$ as a linear combination of $(4, 5)$ and $(1, 3)$.

Solution

$$\alpha_1 (4, 5) + \alpha_2 (1, 3) = (5, 1)$$

$$(\alpha_1 4, \alpha_1 5) + (\alpha_2 + 3\alpha_2) = (5, 1)$$

$$(4\alpha_1 + \alpha_2 , 5\alpha_1 + 3\alpha_2) = (5, 1)$$

$$\Rightarrow \quad 4\alpha_1 + \alpha_2 = 5 \quad \Rightarrow \quad \alpha_2 = 5 - 4\alpha_1$$

$$5\alpha_1 + 3\alpha_2 = 1$$

$$5\alpha_1 + 3(5 - 4\alpha_1) = 1$$

$$5\alpha_1 - 12\alpha_1 + 15 = 1$$

$-7\alpha_1 = -14$

~~the~~ $\alpha_1 = 2$

$\alpha_2 = 5 - 4\alpha_1 = 5 - 4(2) = 5 - 8 = -3$

$\therefore \alpha_1 = 2 \; , \; \alpha_2 = -3$

we write

$$2(4,5) - 3(1,5) = (5,1)$$

<u>Example!</u> ~~Show~~ Determine whether the follow vectors form a basis ~~for~~ for $\mathbb{R}^3$.

$(1,1,1)$, $(1,1,0)$, $(1,0,0)$

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\det(A) = 0 + 0 - 1 = -1 \neq 0$

$\therefore$ ~~there~~ the vectors are linearly independent therefore they form a basis for $\boxed{\mathbb{R}^3}$

$\vec{X} \in \mathbb{R}^2$

$\vec{X} = (x_1, x_2)$

$\vec{X} \in \mathbb{R}^n \Rightarrow \vec{X} = (x_1, x_2, \ldots, x_n)$

<u>Example:</u> $\vec{a} = (1, 0, 4)$, $\vec{b} = (2, -1, 0)$, $\vec{c} = (8, -3, 8)$

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & 0 \\ 8 & -3 & 8 \end{pmatrix} \qquad \det(A) = 0$$

$\Rightarrow$ the vector are not linearly independent

$\therefore$ they donot form a bases for $\mathbb{R}^3$.

SOLVING LINEAR SYSTEMS

The most general form of a linear system of equations is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$

where a_{ij} and c_j are known constants

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}}_{\vec{b}}$$

$$\underline{\underline{A\vec{x} = \vec{b}}}$$

Our goal is to find all values of $x_1, x_2, \ldots, x_n$ that satisfies all the equation.

Solving systems by substitution.

Example:

$$x + 2y + 5z = 2 \quad \text{—— (1)}$$
$$2x - y + 2z = 4 \quad \text{—— (2)}$$
$$3x + y + z = 1 \quad \text{—— (3)}$$

from (1)

$$x = 2 - 2y - 5z \quad \text{—— (4)}$$

Substitute (4) into (2) and (3)

from (2)

$$2(2 - 2y - 5z) - y + 2z = 4$$
$$4 - 4y - 10z - y + 2z = 4$$
$$-5y - 8z = 0$$
$$\Rightarrow y = -\tfrac{8}{5}z \quad \text{—— (5)}$$

from (3)

$$3(2 - 2y - 5z) + y + z = 1$$
$$-5y - 14z = -5$$
$$y = -\tfrac{14}{5}z + 1 \quad \text{—— (6)}$$

equating 5 and 6,

$$-\frac{8}{5}z = -\frac{14}{5}z + 1$$

$$\Rightarrow \quad z = 5/6$$

put z in 5

$$y = -\frac{8}{5}\left(\frac{5}{6}\right) = -\frac{4}{3}$$

put z and y in 4

$$x = 2 - 2\left(-\frac{4}{3}\right) - 5\left(\frac{5}{6}\right)$$

$$x = \frac{1}{2}$$

the solution of the system of equations is

$$\vec{x} = \begin{pmatrix} 1/2 \\ -4/3 \\ 5/6 \end{pmatrix}$$

Gaussian elimination

This technique involves using elementary row operations to rewrite the system of equation into a simpler form that we can then solve by hand.

* Elementary operations.

Consider

$$
(A)\begin{cases} x + 2y + 5z = 2 & \text{--- (1)} \\ 2x - y + 2z = 4 & \text{--- (2)} \\ 3x + y + z = 1 & \text{--- (3)} \end{cases}
$$

- multiplication of a row (equation) by a non-zero number.

Eg multiply row (1) by 2

$$
(B)\begin{cases} 2x + 4y + 10z = 4 \\ 2x - y + 2z = 4 \\ 3x + y + z = 1 \end{cases}
$$

this system has the same solution has system (A).

— Adding a multiple of one row to another row

Eg for system (B) add rows (1) and (2)

$$2x + 4y + 10z = 4$$
$$4x + 3y + 12z = 8$$
$$3x + y + z = 1$$

— interchange two rows

<u>Example!</u> interchange rows (1) and (3).

$$3x + y + z = 1$$
$$4x + 3y + 12z = 8$$
$$2x + 4y + 10z = 4$$

note:
elementary row operations do not change the solution of the system.

Augmented matrices

consider the system

$$x + 4y + 2z = 6$$
$$3x + y + 3z = 4$$
$$x + 3y + z = 2$$

the coefficient matrix of the system is

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$$

the augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 6 \\ 3 & 1 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{array}\right)$$

Gaussian elimination method

Consider

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

the solution of this system is

$$x_1 = 2, \quad x_2 = 4, \quad x_3 = 3$$

The matrix is in <u>reduced row echelon form</u>

Consider

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$\boxed{\text{The matrix is in row echelon form}}$

a matrix in this form is called <u>an upper triangular matrix</u>

$$\left(\begin{array}{ccc|} x & & \\ x & x & \\ x & x & x \end{array}\right)$$

<u>lower triangular matrix</u>

Let us solve the system in ①.

from row ③

$$x_3 = 3$$

from row ②

$$-x_2 + 2x_3 = 4 \Rightarrow x_2 = 2x_3 - 4$$

$$= 6 - 4$$

$$x_2 = 2$$

from row ①

$$x_1 + 3x_2 + 2x_3 = 2$$

$$x_1 = 2 - 3x_2 - 2x_3$$

$$x_1 = 2 - 6 - 6 = -10$$

∴ the solution is

$$\begin{pmatrix} -10 \\ 2 \\ 3 \end{pmatrix}$$

~~this process~~

the process of solving the upper triangular system is called <u>Back substitution</u>.

from row 1 to,

$$x - y + z = 2$$

$$x = 2 + y - z = 2 + \tfrac{1}{2} - \tfrac{15}{4} = -\tfrac{5}{4}$$

$\therefore$ the solution of the system is

$$\begin{pmatrix} -5/4 \\ 1/2 \\ 15/4 \end{pmatrix}$$

Example: consider

$$x - y + z = 2$$
$$3x - 2y + z = -1$$
$$x + y + z = 3$$

the augment matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & -1 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 = R_2 - 3R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 4 & 15 \end{array}\right) \quad R_3 = R_3 - 2R_2$$

The matrix is now in row echelon form.

Let us solve the system.

row 3 $\Rightarrow$ $4z = 15$ $\Rightarrow z = 15/4$

row 2 $\Rightarrow$ $y - 2z = -7$ $\Rightarrow y = -7 + 2z = -7 + 2\left(\frac{15}{4}\right)$

$y = \frac{1}{2}$

Example:

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 1 & 2 & -1 & -5 & -18 \\ 0 & 3 & 0 & -3 & -6 \\ -1 & 4 & 1 & 1 & 14 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 6 & -1 & -6 & -15 \end{array}\right) \quad \begin{array}{l} \\ R_2 = R_2 - R_1 \\ \\ R_4 = R_4 + R_1 \end{array}$$

interchange rows (2) and (3)

$$* \left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 6 & -1 & -6 & -15 \end{array}\right) \quad R_4 = R_4 - 6R_2$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & -1 & 0 & -3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right) \quad R_4 = R_4 + R_3$$

from row 4, $x_4 = 4$

from row 3, $z = 3$

from row 2 $y = 2$

and lastly $x = 1$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$