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Last day

$$b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n = 0$$
$$b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n = 0$$
$$\vdots$$
$$b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n = 0$$

this is a homogeneous system of equations.

* proproperties

- - unique solution (trivial solution)
  - infinitely many solutions.
- if $\vec{X}$ and $\vec{Y}$ are solutions then $\vec{X} + \vec{Y}$ is also a solution
- If $\vec{X}$ is a solution, then $\alpha\vec{X}$ is also a solution, $\alpha \in \mathbb{R}$.
- if $\vec{X}$ and $\vec{Y}$ are solutions then the linear combination of $\vec{X}$ and $\vec{Y}$ is also a solution

i.e $\alpha_1\vec{X} + \alpha_2\vec{Y}$ is a solution

Connection of solutions ~~of~~ to homogeneous and inhomogeneous systems.

Suppose we have ~~the in hom~~

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n &= c_1 \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n &= c_2 \\ \vdots \qquad \vdots \qquad & \;\;\vdots \\ b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n &= c_m \end{aligned}\right\}$$

and the solution of the ~~sys~~ system is

$$\vec{X} = \vec{q} + \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_n\vec{a}_n$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are parameter.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then $\vec{X} = \vec{q}$ is also a solution of the ~~in~~ system. And the linear combination

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_n\vec{a}_n$$ is the

general solution of the corresponding homogeneous system of equation.

Example:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ -1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 1 & 14 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & 2 & 1 & 14 \\ 0 & 1 & 2 & 1 & 14 \end{array}\right) \quad R_2 = R_2 + R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & 2 & 1 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 = R_3 - R_2$$

the system has infinitely many solutions.

rank = 2

from row 2, $x_2 + 2x_3 + x_4 = 14$

$$x_2 = 14 - 2x_3 - x_4$$

from row 1, $x_1 + x_3 = 10$

$$x_1 = 10 - x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 - x_3 \\ 14 - 2x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 10 \\ 14 \\ 0 \\ 0 \end{pmatrix}}_{\vec{q}} + x_3 \underbrace{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{a}_1} + x_4 \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}_2}$$

Check

- Let $X_3 = X_4 = 0$

then $\vec{x} = \begin{pmatrix} 10 \\ 14 \\ 0 \\ 0 \end{pmatrix} = \vec{q}$

$x_1 = 10$
$x_2 = 14$
$x_3 = 0$
$x_4 = 0$

row 1, $x_1 + x_3 = 10$

$10 = 10$ ✓

row 2, $-x_1 + x_2 + \cancel{x_3}^{0} + \cancel{x_4}^{0} = 4$

$-10 + 14 = 4$ ✓

row 3, $x_2 + \cancel{2x_3}^{0} + \cancel{x_4}^{0} = 14$

$14 = 14$ ✓

- the general solution of the correspond homogeneous system is

$$X_3 \vec{a_1} + X_4 \vec{a_2} \quad , X_3, X_4 \in \mathbb{R}$$

Take $X_3 = 1$, $X_4 = 1$

Check that $\vec{Y} = \begin{pmatrix} -1 \\ -3 \\ 1 \\ 1 \end{pmatrix}$ satisfies the homogeneous system.

GEOMETRIC APPLICATIONS

Recall, that if we have vectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ the linear combination of these vectors is

$$LC = \alpha_1 \vec{X}_1 + \alpha_2 \vec{X}_2 + \cdots + \alpha_n \vec{X}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ ~~[illegible]~~

If $\exists$ $\alpha_1, \alpha_2, \ldots, \alpha_n$ that are not all zeros such that

$$LC = \vec{0}$$

then $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ are linearly dependent

Now,

$$LC = \vec{0}$$

$$\Rightarrow \alpha_1 \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1m} \end{pmatrix} + \alpha_2 \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2m} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\alpha_1 x_{11} + \alpha_2 x_{21} + \cdots + \alpha_n x_{n1} &= 0 \\
\alpha_1 x_{12} + \alpha_2 x_{22} + \cdots + \alpha_n x_{n2} &= 0 \\
\vdots \qquad \qquad \vdots \qquad \qquad & \vdots \quad \vdots \\
\alpha_1 x_{1m} + \alpha_2 x_{2m} + \cdots + \alpha_n x_{nm} &= 0
\end{aligned}$$

Therefore, we have a homogeneous system whose solution is $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

~~and~~. If the only solution of the system is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then the vectors $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$ are linearly independent

Otherwise the are linearly dependent.

Example! Determine whether the following vectors are linearly independent

$\vec{a}_1 = (1, 2, 0)$, $\vec{a}_2 = (1, 1, 1)$, $\vec{a}_3 = (1, 2, 1)$

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = 0 \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$2\alpha_1 + \alpha_2 + 2\alpha_3 = 0$$

$$\alpha_2 + \alpha_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad R_2 = R_2 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\alpha_3 = 0$

$-\alpha_2 = 0$

$\alpha_2 = 0$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$\Rightarrow \alpha_1 = 0$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ The vectors are linearly independent.

$\Rightarrow$ The form a bases for $\mathbb{R}^3$

i.e. $\vec{x} \in \mathbb{R}^3$

$$\vec{x} = \beta_1 \vec{a}_1 + \beta_2 \vec{a}_2 + \beta_3 \vec{a}_3$$

where $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$.

<u>Example:</u> Find the intersection of the following planes.

$$\left.\begin{aligned} x_1 - x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 + x_3 &= 2 \\ -x_1 + 2x_2 + x_3 &= 1 \end{aligned}\right\}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 2 & 2 & 1 & 2 \\ -1 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 4 & -7 & 0 \\ 0 & 1 & 5 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 4 & -7 & 0 \\ 0 & 0 & \cancel{18}\ 27 & 8 \end{array}\right) R_3 \to 4R_3 - R_2$$

The ~~[illegible]~~ system has a unique solution which is the point of intersection of the planes.

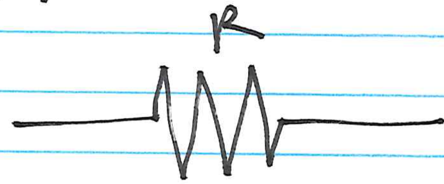
RESISTOR NETWORKS

Ohm's law

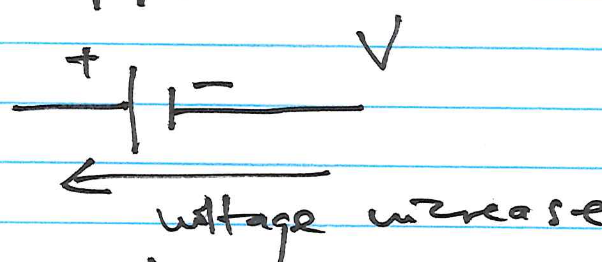
The potential difference (voltage) across an ideal conductor is proportional to the current through it. The constant of proportionality is called the resistance, R.

$$V = IR.$$

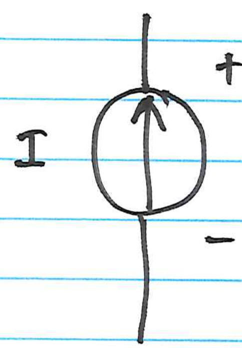
- resistors



- voltage source



- Current source



Kirchhoff

Basic problem:

Find the currents through each resistor and each power source, and also the voltage drops across each current source.

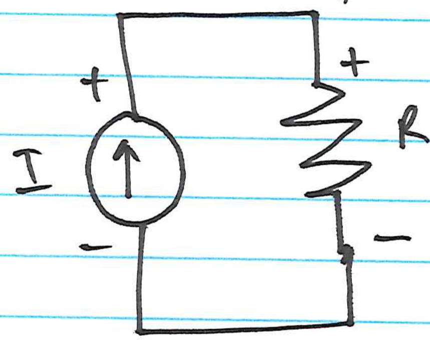
fundamental problem:

A subproblem is to find the current through every power source and the voltage drop across each current source.

Kirchhoff's laws:

- The sum of voltage drops around any closed loops in the network must be zero
- The sum of currents entering a node must be zero, i.e the sum of currents flowing into a node is equal to the sum of currents flowing out of the node.

Example: Consider the simple network.

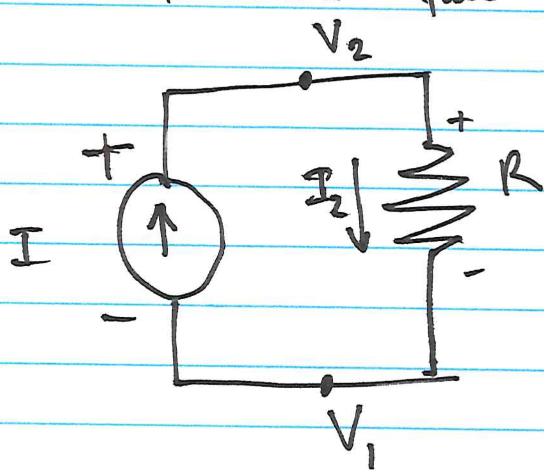


Suppose current I and resistance R are known.

$$V = IR.$$

is the voltage across the resistor.

Let us introduce two nodes into the network



We want find V_1, V_2, and I_2.

Ohm's law

$$\Delta V = V_2 - V_1 = I_2 R$$

$$\Rightarrow \quad V_2 - V_1 - I_2 R = 0$$

Kirchhoff's second:

<u>node 2:</u>

$$I = I_2 \Rightarrow I - I_2 = 0$$

<u>node 1:</u>

$$I_2 = I \Rightarrow I_2 - I_1 = 0$$

$\therefore$ we have

system of equations:

$$\begin{cases} V_2 - V_1 - I_2 R = 0 \Rightarrow -I_2 R - V_1 + V_2 = 0 \\ I_2 = I_1 \\ I_2 = I \end{cases}$$

$$\begin{pmatrix} -R & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ I \\ I_1 \end{pmatrix}$$

$r_2 = R r_2 + r_1$

$$\left(\begin{array}{ccc|c} -R & -1 & +1 & 0 \\ 1 & 0 & 0 & I \\ 1 & 0 & 0 & I \end{array}\right) \sim \left(\begin{array}{ccc|c} -R & -1 & 1 & 0 \\ 0 & -1 & 1 & IR \\ 0 & -1 & 1 & IR \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -R & -1 & 1 & 0 \\ 0 & -1 & 1 & IR \\ 0 & 0 & 0 & 0 \end{array}\right)$$

from row 2, $-V_1 + V_2 = IR$

$$\Rightarrow \quad V_2 = IR + V_1$$

from row 1,

$$-RI_2 - V_1 + V_2 = 0$$

$$\Rightarrow \quad V_1 = V_2 - RI_2$$

$$\cancel{V_1} = IR + \cancel{V_1} - RI_2$$

$$\left.\begin{aligned} I_2 &= I \\ \Rightarrow \quad V_2 &= IR + V_1 \end{aligned}\right\}$$

let $V_1 = t$, $t \in \mathbb{R}$.

$$\Rightarrow \quad I_2 = I$$

$$V_1 = t$$

$$V_2 = IR + t$$

$$\begin{pmatrix} I_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} I \\ t \\ IR + t \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ IR \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \quad V_2 - V_1 = (IR + t) - t = IR$

and $\quad \underline{I_2 = I}$

Additional note (Review?)

Multiplying a ~~vector~~ matrix by a ~~matrix~~ vector

Suppose we have a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and we want to multiply by a vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

multiply each row by the vector

then we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}X_3 \\ a_{21}x_1 + a_{22}X_2 + a_{23}X_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

So for our system

$$\left.\begin{aligned} -RI_2 - V_1 + V_2 &= 0 \\ I_2 &= I \\ I_2 &= I \end{aligned}\right\} \text{——} (*)$$

we can write it using the idea of matrix multiplication as

$$\begin{pmatrix} -R & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

expand this using matrix multiplication and you will get (*)