Last day

* **Random walks**

If P is the matrix of a random walk that has initial state $\vec{X}_0$, then the state of the network after k time-steps is given by

$$\vec{X}_k = P^k \vec{X}_0$$

- the sum of the entries of each column of P must sum to 1.

- the sum of all entries of each vector $\vec{X}_n$ for $n = 0, 1, 2, \ldots, k$ must ~~sum~~ equal ~~to~~ 1.

– the ~~transp~~ transpose of a matrix

If $A = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 4 & 6 \end{pmatrix}$

and if A is an $n \times n$ matrix with $A^T = A$, then A is a symmetric matrix.

– inverse of a matrix

If A is an $n \times n$ matrix with inverse A^{-1}, then $A^{-1}A = AA^{-1} = I$

where I is the identity matrix.

Computing the inverse of a matrix.

Let A be an $n \times n$ matrix that is invertible, and let $B = A^{-1}$.

Then

$$AB = BA = I$$

(*) $AB = I$

If $\cancel{A = \begin{pmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_n \\ | & | & & | \end{pmatrix}}$, $B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix}$

$$A = \begin{pmatrix} \text{---} & A_1 & \text{---} \\ \text{---} & A_2 & \text{---} \\ & \vdots & \\ \text{---} & A_n & \text{---} \end{pmatrix}$$

$$AB = \begin{pmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_n \\ | & | & & | \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix}$$

from (*),

$$\begin{pmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} Ab_1 &= e_1 \\ Ab_2 &= e_2 \\ &\vdots \\ Ab_n &= e_n \end{aligned}$$

We have n systems of equations where the solution of ~~each~~ each system is a column of matrix B (in the same order).

<u>Example:</u> Suppose we want to find the inverse of matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

$\Rightarrow$ we have 2 systems of equations to solve

Let $B = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix}$ be A^{-1}.

Then

$$Ab_1 = e_1$$

$$Ab_2 = e_2$$

$b_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$b_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

The first system is

$$\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 4 & 1 \\ 3 & 2 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 2 & 4 & 1 \\ 3 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1/2 \\ 3 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 1/2 \\ 0 & -4 & -3/2 \end{array}\right) \quad R_2 = R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 1/2 \\ 0 & 1 & +3/8 \end{array}\right) \quad R_2 = \frac{R_2}{-4}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -1/4 \\ 0 & 1 & +3/8 \end{array}\right) \quad R_1 = R_1 - 2R_2$$

$$\Rightarrow \quad b_1 = \begin{pmatrix} -1/4 \\ 3/8 \end{pmatrix}$$

For the ~~sty~~ second system,

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -4 & 1 \end{array}\right) \quad R_2 = R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/4 \end{array}\right) \quad R_2 = \frac{R_2}{-4}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \end{array}\right) \quad \begin{array}{l} R_1 = R_1 - 2R_2 \\ \text{RREF} \end{array}$$

$$b_2 = \begin{pmatrix} 1/2 \\ -1/4 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -1/4 & 1/2 \\ 3/8 & -1/4 \end{pmatrix}$$

We observed that we used the same set of elementary row operations for the 2 systems. Which implies if we have an $n \times n$ matrix, we need to carry out this operations n times to get the inverse of the matrix.

Instead of doing this, let us use the <u>super-augmented matrix</u>.

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & -4 & -3/2 & 1 \end{array}\right) \quad R_2 = R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & +\frac{3}{8} & -\frac{1}{4} \end{array}\right) \quad R_2 = \frac{R_2}{-4}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{3}{8} & -\frac{1}{4} \end{array}\right) \quad R_1 = R_1 - 2R_2$$

$$\underbrace{\begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & -\frac{1}{4} \end{pmatrix}}_{A^{-1}}$$

This technique can be applied to a matrix of any dimension.

<u>Outline of the procedure</u>

- Given an invertible $n \times n$ matrix A
- construct the super-augmented matrix

$$[A \mid I]$$

- reduce the matrix to ~~the reduced~~ the form

$$[I \mid B]$$

- then $B = A^{-1}$.

consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = ad - bc.$$

using the super-augmented matrix technique, we can show that

$$A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if $\det(A) = 0$, then we have $\frac{1}{0}$, undefined

$\Rightarrow$ A^{-1} does not exist.

Inverse of product of matrices.

Let A and B be invertible matrices,

Then $$(AB)^{-1} = B^{-1}A^{-1}$$

Check!

H and D are matrices.

If $H^{-1} = D$, then $H D = H H^{-1} = I$

$\Rightarrow$ $\cancel{(AB)^{-1}(B^{-1}A^{-1})}$ should give I.

$$(AB)^{-1}(AB) = (B^{-1}A^{-1})(AB)$$
$$= B^{-1}\left(\underline{A^{-1}A}\right)B$$
$$= B^{-1}(I B)$$
$$= B^{-1}B = I$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

Example: Determine which of these matrices are invertible.

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 3 \\ -1 & -1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

Not invertible (singular) — Invertible

Def: singular matrix

An $n \times n$ matrix that is not invertible is a singular matrix.

- A square matrix is singular if and only if its determinant is zero.

DETERMINANTS

Recall that if we have a 2x2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(A) = a_{11} a_{22} - a_{12} a_{21}$$

and for a 3x3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

this is tedious if $n \geq 4$!

Consider a 2x2 upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

$$\det(A) = a_{11} a_{22} - 0 \cdot a_{12} = a_{11} a_{22}$$

for a lower triangular matrix,

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}, \quad \det(A) = a_{11} a_{22} - 0 \cdot a_{21}$$

$$= a_{11} a_{22}$$

For a 3×3 ~~upper~~ lower triangular matrix,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{11} \left(a_{22} a_{33} - 0 \right) - 0 + 0$$

$$= a_{11} a_{22} a_{33}$$

Similarly, for an upper triangular 3×3 matrix,

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}$$

Observe that the determinant of an upper or lower triangular matrix is the product of the diagonal entries.

Therefore, if we have an $n \times n$ matrix, we can reduce it to an upper or lower triangular form using elementary row operation, and then compute the determinant by multiplying the diagonal entries.

However, some elementary row operations ~~do~~ change the determinant of a matrix.

Let A be an $n \times n$ matrix:

① If B is obtained from A by multiplying one row of A by a constant α, then

$$\det(B) = \alpha \det(A).$$

e.g. $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3/2 \\ 4 & 5 \end{pmatrix}$

$\det(A) = -2$, $\det(B) = -1$

$\Rightarrow \det(B) = \frac{1}{2} \cdot \det(A)$.

② If B is obtained ~~b~~ from A by switching two rows of A, then $\det(B) = -\det(A)$.

③ If B is obtained from A by adding a multiple of one row to another then

$$\det(B) = \det(A).$$

(4) $\det(A) = 0$ if and only if A is not invertible.

(5) If A and B are two matrices of the same dimensions, then

$$\det(AB) = \det(A)\det(B).$$

(6) $\det(A^T) = \det(A)$.

Example: Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & 3 & 9 \\ 0 & -1 & 3 & -7 \\ 0 & 2 & 0 & 2 \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$= 3\det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 3 & -7 \\ 0 & 2 & 0 & 2 \end{pmatrix} \quad R_2 = R_2/3$$

$$= 3\det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & -2 & -4 \end{pmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$= (4\times 3)\det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & -4 \end{pmatrix} \quad R_3/4$$

$$= 12 \det \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -6 \end{pmatrix} \quad R_4 + 2R_3$$

$$\therefore \det(A) = 12\,(1)(1)(1)(-6)$$

$$= \underline{\underline{-72}}$$