Last day

* application of eigenanalysis to D.E.

— system of linear differential equations.

e.g.

$$x_1'(t) = a_{11} x_1(t) + a_{12} x_2(t) + a_{13} x_3(t)$$

$$x_2'(t) = a_{21} x_1(t) + a_{22} x_2(t) + a_{23} x_3(t)$$

$$x_3'(t) = a_{31} x_1(t) + a_{32} x_2(t) + a_{33} x_3(t).$$

this system can be written in matrix form as

$$\vec{X}'(t) = A\vec{X}(t)$$

where $\vec{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The general solution of this system is of the form

$$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3$$

where λ_1, λ_2 and λ_3 are eigenvalues of A with corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. c_1, c_2, c_3 are constants.

If given an initial condition $\vec{X}(0) = X_0$ then we need to solve the system

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 + c_3 \vec{V}_3 = \vec{X}_0$$

to get the constants c_1, c_2, and c_3.

<u>Example:</u> Solve

$$\begin{aligned} y_1'(t) &= y_1 + 2y_3, & y_1(0) &= 0 \\ y_2'(t) &= 2y_2, & y_2(0) &= 1 \\ y_3'(t) &= 2y_1 + y_2 + y_3, & y_3(0) &= 1 \end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

Let λ be an eigenvalue of A, then

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(1-\lambda) - 0 + 2\left(-2(2-\lambda)\right) = 0$$

$\vdots$

$$(2-\lambda)(\lambda+1)(\lambda-3) = 0$$

$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$

For $\lambda_1 = -1, \qquad (A - \lambda_1 I)\vec{V}_1 = \vec{0}$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow V_2 = 0 \quad , V_1 = -V_3.$

take $V_3 = 1$, $V_1 = -1$

then $\vec{V}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$, $(A - \lambda_2 I)\vec{V}_2 = \vec{0}$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-1+4$

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v_2 = -3v_3, \quad v_1 = 2v_3$$

take $v_3 = 1$, $v_2 = -3$, $v_1 = 2$

$$\therefore \vec{V}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 3$, $(A - \lambda_3 I)\vec{V}_3 = \vec{0}$

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The general solution of the system is

$$\vec{Y} = c_1 e^{\lambda_1 t}\vec{V}_1 + c_2 e^{\lambda_2 t}\vec{V}_2 + c_3 e^{\lambda_3 t}\vec{V}_3$$

$$= c_1 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

~~As~~ we are given

$$y_1(0) = 0, \quad y_2(0) = 1, \quad y_3(0) = 1$$

$$\Rightarrow \vec{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

applying the initial condition,

$$c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 = \tfrac{1}{3}, \quad c_2 = -\tfrac{1}{3}, \quad c_3 = 1$$

$$\Rightarrow \vec{Y}(t) = \frac{1}{3}e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}e^{2t}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

In component form, we have

$$y_1(t) = -\tfrac{1}{3}e^{-t} - \tfrac{2}{3}e^{2t} + e^{3t}$$

$$y_2(t) = e^{2t}$$

$$y_3(t) = \tfrac{1}{3}e^{-t} - \tfrac{1}{3}e^{2t} + e^{3t}$$

Aside

Suppose you have an eigenvalue that is repeated, for instance $\lambda_1 = 1$ (twice) and ~~you~~ you get only one eigenvector, then we need use the generalized eigenvector method to find the other ~~eq~~ linearly independent eigenvector.

If $\vec{v}_1$ is the ~~e~~ eigenvector obtained,

$$(A - \lambda_1 I)^2 \vec{v}_2 = \vec{0}$$

where $\vec{v}_2$ is the second linearly independent eigenvector.

Example: (complex eigenvalues and eigenvectors)

Find the general solution of the system

$$\vec{X}'(t) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \vec{X}(t)$$

and find $\vec{X}(t)$ that satisfies $\vec{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

let $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

let λ be eigenvalue of A, then

$$|A - \lambda I| = 0.$$

$$\left| \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm\sqrt{-4} = \pm\sqrt{(-1)\times 4} = \pm 2\sqrt{-1} = \pm 2i$$

$$\Rightarrow \quad \lambda_1 = 2i \quad , \quad \lambda_2 = -2i$$

For $\lambda_1 = 2i$, $(A - \lambda_1 I)\vec{V}_1 = \vec{0}$

$$\Rightarrow \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2i & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-2i V_1 = 2V_2$$

$$V_1 = \frac{2V_2}{-2i} = \frac{V_2}{-i} \frac{\times i}{\times i} = iV_2$$

take $V_2 = 1$, $V_1 = i$

$$\vec{V}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

For $\lambda_2 = -2i$, the eigenvector is complex conjugate of $\vec{V}_1$,

$$\Rightarrow \vec{V}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$\Rightarrow$ the general solution of the system is

$$\vec{X}(t) = c_1 e^{2it} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-i2t} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

observe that we have a complex solution. Usually, ~~that~~ often we want real solutions, therefore, we need to put the solution in to real-valued form.

The real-valued solution is of the form

$$\vec{X}(t) = \alpha_1 \, Re\left(\vec{v}_1 e^{\lambda_1 t}\right) + \alpha_2 \, Im\left(\vec{v}_1 e^{\lambda_1 t}\right)$$

take $\vec{v}_1 e^{\lambda_1 t} = e^{i2t} \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$e^{i2t} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} \left(\cos(2t) + i \sin(2t)\right)$$

$$= \begin{pmatrix} i\cos(2t) - \sin(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}$$

$$e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + i\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

$$\operatorname{Re}\left(e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}$$

$$\operatorname{Im}\left(e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

take $\vec{v}_2 e^{\lambda_2 t} = e^{-i2t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$e^{-i2t}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}\left(\cos(2t) - i\sin(2t)\right)$$

$$= \begin{pmatrix} -i\cos(2t) - \sin(2t) \\ \cos(2t) - i\sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} - i\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

$$\operatorname{Re}\left(e^{-i2t}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}, \quad \operatorname{Im}\left(e^{-i2t}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

$\Rightarrow$ ~~the real-valued~~

the general real-valued solution is

$$\vec{X}(t) = \alpha_1 \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

α_1, α_2 are constants

$$\vec{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\alpha_1 \begin{pmatrix} -\sin(0) \\ \cos(0) \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\alpha_1 = -1, \alpha_2 = 1.$

$\therefore$ the real-valued solution that satisfies $\vec{X}(0)$ is

$$\vec{X}(t) = (-1) \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}.$$

An alternative approach to get the ~~###~~ real-valued solution:

we have

$$\vec{X}(t) = c_1 e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-i2t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Let us apply the initial condition directly to this solution.

$$\Rightarrow \quad c_1\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad \left(\begin{array}{cc|c} i & -i & 1 \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & i \\ 1 & 1 & -1 \end{array}\right) iR_1$$

$$\sim \left(\begin{array}{cc|c} -1 & 1 & i \\ 0 & 2 & -1+i \end{array}\right) \quad R_2 = R_2 + R_1$$

$$\Rightarrow \quad c_2 = \frac{-1+i}{2}$$

$$-c_1 + c_2 = i \quad \Rightarrow \quad ~~###~~ \quad c_1 = c_2 - i$$

$$\Rightarrow \quad c_1 = -\frac{(1+i)}{2}$$

$$\vec{X}(t) = -\frac{(1+i)}{2} e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{(-1+i)}{2} e^{-2it}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

~~We need to simplify~~

Let us write this solution in a better way,

$$\vec{X}(t) = \underbrace{\frac{(-1-i)}{2} e^{i2t} \begin{pmatrix} i \\ 1 \end{pmatrix}}_{z} + \underbrace{\frac{(-1+i)}{2} e^{-2it} \begin{pmatrix} -i \\ 1 \end{pmatrix}}_{\bar{z}}$$

Observe that the two terms in the equation are complex conjugate of each other.

Recall, if $z = x+iy$, $\bar{z} = x-iy$

$$z + \bar{z} = 2x = 2\,Re(z)$$

$$\therefore \vec{X}(t) = 2\,Re(z) = 2\,Re\left[\frac{(-1-i)}{2} e^{i2t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right]$$

What is left is for us to find $Re\left[\frac{(-1-i)}{2} e^{i2t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right]$

Now,

$$\frac{(-1-i)}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{i2t} = \left(\frac{-1-i}{2}\right)\begin{pmatrix} i \\ 1 \end{pmatrix}\left(\cos(2t) + i\sin(2t)\right)$$

$$= \left(-\frac{1}{2} - \frac{i}{2}\right)\begin{pmatrix} i\cos(2t) - \sin(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} - \frac{i}{2}\begin{pmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}$$

$$\frac{(-1-i)}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{i2t} = -\frac{1}{2}\begin{pmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} - \frac{i}{2}\begin{pmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sin(2t) - i\cos(2t) \\ -\cos(2t) - i\sin(2t) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} i\sin(2t) + \cos(2t) \\ -i\cos(2t) + \sin(2t) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sin(2t) + \cos(2t) \\ -\cos(2t) + \sin(2t) \end{pmatrix} + \frac{i}{2}\begin{pmatrix} \sin(2t) - \cos(2t) \\ -\sin(2t) - \cos(2t) \end{pmatrix}$$

$$\therefore \mathrm{Re}\left[\frac{(-1-i)}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{i2t}\right] = \frac{1}{2}\begin{pmatrix} \sin(2t) + \cos(2t) \\ -\cos(2t) + \sin(2t) \end{pmatrix}$$

But

$$\vec{X}(t) = 2\,\mathrm{Re}\left[\frac{(-1-i)}{2} e^{i2t}\begin{pmatrix} i \\ 1 \end{pmatrix}\right]$$

$$= 2\left[\frac{1}{2}\begin{pmatrix} \sin(2t) + \cos(2t) \\ -\cos(2t) + \sin(2t) \end{pmatrix}\right]$$

$$\vec{X}(t) = \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix} + \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

This is the real-valued solution.

Observe that this is exactly the same as the one we obtained earlier.

Example: Solve

$$\vec{X}'(t) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \vec{X}(t)$$

with Initial condition $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

Let λ be an eigenvalue of A, then

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) + 1 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

using the quadratic formula,

$$\lambda = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$\lambda_1 = 2+i, \quad \lambda_2 = 2-i.$$

For $\lambda_1 = 2+i$, $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$

$$\Rightarrow \begin{pmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right] \overset{iR_1}{\sim} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad v_1 = -i v_2$,

take $v_2 = 1$, then $v_1 = -i$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2-i$, $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

therefore,

$$\vec{X}(t) = c_1 e^{(2+i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2 e^{(2-i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\vec{X}(t) = c_1 e^{2t} e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2 e^{2t} e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= e^{2t} \left[c_1 e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix} \right]$$

~~now~~ let us find the real and imaginary part of $e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \left(\cos(t) + i \sin(t) \right)$$

$$= \begin{pmatrix} -i \cos(t) + \sin(t) \\ \cos(t) + i \sin(t) \end{pmatrix}$$

$$= \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} + i \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$$

$$\mathrm{Re}\left(e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

$$\mathrm{Im}\left(e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$$

$\therefore$ the general real-valued solution is

$$\vec{X}(t) = \alpha_1 e^{2t}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} + \alpha_2 e^{2t}\begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$$

Let us apply the initial condition

$$\vec{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \alpha_1 \begin{pmatrix} \sin(0) \\ \cos(0) \end{pmatrix} + \alpha_2 \begin{pmatrix} -\cos(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \alpha_1 = +1, \quad \alpha_2 = -1$$

$$\therefore \vec{X}(t) = e^{2t}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} - e^{2t}\begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$$