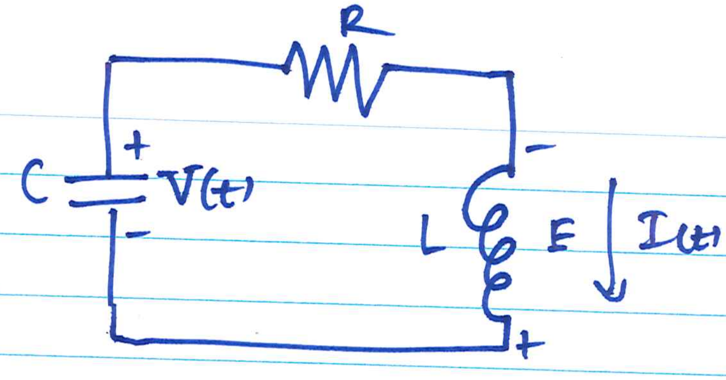
Last day

* LCR circuits



* capacitors

- ~~sources~~ serves as source of voltage

$$\frac{dV(t)}{dt} = -\frac{i}{C}$$, where C – capacitance

* inductors

- serves as source of current

$$\frac{dI(t)}{dt} = -\frac{v}{L}$$, L - inductance.

* usually we want to know how V(t) and I(t) evolve in time

Given a circuit

- derive differential equations for V(t) and I(t)
- use the equations to construct a system of equations
- use eigen-analysis to solve the system

- interprete the dynamics of V(t) and I(t) using the eigenvalues of the system.

Revision

Example: Find the real-valued general solution of the system given below

$$\vec{X}'(t) = A\vec{X}(t)$$

where $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

For the eigenvalues,

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2\right) + 1\left(-1(1-\lambda)\right) + 2\left(+1(1-\lambda)\right) = 0$$

$$(1-\lambda)\left((1-\lambda)^2 - 1 + 2\right) = 0$$

$$(1-\lambda)\left(\lambda^2 - 2\lambda + 1 - 1 + 2\right) = 0$$

$$(1-\lambda)(\lambda^2-2\lambda+2)=0$$

$\lambda_1 = 1$, $\lambda^2 - 2\lambda + 2 = 0$

$$\lambda_{2,3} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\lambda_2 = 1+i$

$\lambda_3 = 1-i$

For $\lambda_1 = 1$, $(A-\lambda_1 I)\vec{V}_1 = \vec{0}$

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_2 = -2x_3$, $x_2 = 2x_3$

$x_1 = 0$

take $x_3 = 1$,

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1+i$, $(A-\lambda_2 I)\vec{V_2} = \vec{0}$

$$\Rightarrow \begin{pmatrix} -i & -1 & 2 \\ -1 & -i & 0 \\ -1 & 0 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right] \; iR_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & -2i & 2i & 0 \\ 0 & -i & i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-x_2 = -x_3 \quad \Rightarrow \quad x_2 = x_3$$

$$x_1 - ix_2 + 2ix_3 = 0$$

$$x_1 = -2ix_3 + ix_2 = -2ix_3 + ix_3 = -ix_3$$

take $x_3 = 1$, $x_1 = -i$, $x_2 = 1$

$$\vec{V_2} = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1-i$, $\vec{V_3} = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ The general solution of the system is

$$\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{V}_1 + c_2 e^{\lambda_2 t}\vec{V}_2 + c_3 e^{\lambda_3 t}\vec{V}_3$$

$$= c_1 e^{t}\begin{pmatrix}0\\2\\1\end{pmatrix} + c_2 e^{(1+i)t}\begin{pmatrix}-i\\1\\1\end{pmatrix} + c_3 e^{(1-i)t}\begin{pmatrix}i\\1\\1\end{pmatrix}$$

Consider

$$c_2 e^{(1+i)t}\begin{pmatrix}-i\\1\\1\end{pmatrix} + c_3 e^{(1-i)t}\begin{pmatrix}i\\1\\1\end{pmatrix}$$

$$= e^{t}\left[c_2\begin{pmatrix}-i\\1\\1\end{pmatrix}e^{it} + c_3 e^{-it}\begin{pmatrix}i\\1\\1\end{pmatrix}\right]$$

$$= e^{t}\left[c_2\begin{pmatrix}-i\\1\\1\end{pmatrix}(\cos(t)+i\sin(t)) + c_3\begin{pmatrix}i\\1\\1\end{pmatrix}(\cos(t)-i\sin(t))\right]$$

$$= e^{t}\begin{bmatrix}-ic_2\cos(t) - i^2 c_2\sin(t) + c_3 i\cos(t) - i^2 c_3\sin(t)\\ c_2\cos(t) + ic_2\sin(t) + c_3\cos(t) - ic_3\sin(t)\\ c_2\cos(t) + ic_2\sin(t) + c_3\cos(t) - ic_3\sin(t)\end{bmatrix}$$

$$= e^{t}\begin{bmatrix}(c_2+c_3)\sin(t) + i(c_3-c_2)\cos(t)\\ (c_2+c_3)\cos(t) - i(c_3-c_2)\sin(t)\\ (c_2+c_3)\cos(t) - i(c_3-c_2)\sin(t)\end{bmatrix}$$

$$= e^{t}\left[(c_2+c_3)\begin{pmatrix}\sin(t)\\ \cos(t)\\ \cos(t)\end{pmatrix} + i(c_3-c_2)\begin{pmatrix}\cos(t)\\ -\sin(t)\\ -\sin(t)\end{pmatrix}\right]$$

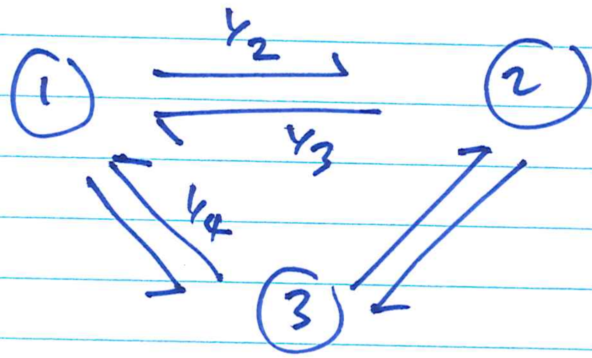
let $c_2+c_3=\alpha_2$, $i(c_3-c_2)=\alpha_3$

$$= e^{t}\left[\alpha_2\begin{pmatrix}\sin(t)\\ \cos(t)\\ \cos(t)\end{pmatrix} + \alpha_3\begin{pmatrix}\cos(t)\\ -\sin(t)\\ -\sin(t)\end{pmatrix}\right]$$

$\therefore$ the real-valued general solution is

$$\vec{X}(t) = \alpha_1 e^{t}\begin{pmatrix}0\\ 2\\ 1\end{pmatrix} + \alpha_2 e^{t}\begin{pmatrix}\sin(t)\\ \cos(t)\\ \cos(t)\end{pmatrix} + \alpha_3 e^{t}\begin{pmatrix}\cos(t)\\ -\sin(t)\\ -\sin(t)\end{pmatrix}$$

Example! consider the random walk below



(a) construct the transition matrix P

(b) find $\vec{X}_n$ as $n \to \infty$.

(a)

$$P = \begin{pmatrix} 0 & 1/3 & 1/4 \\ 1/2 & 0 & 3/4 \\ 1/2 & 2/3 & 0 \end{pmatrix}$$

(b) If λ is an eigenvalue of P, then

$$|P - \lambda I| = 0.$$

$$\begin{vmatrix} -\lambda & 1/3 & 1/4 \\ 1/2 & -\lambda & 3/4 \\ 1/2 & 2/3 & -\lambda \end{vmatrix} = 0$$

$$-\lambda\left(\lambda^2 - \tfrac{1}{2}\right) - \tfrac{1}{3}\left(-\tfrac{\lambda}{2} - \tfrac{3}{8}\right) + \tfrac{1}{4}\left(\tfrac{1}{3} + \tfrac{\lambda}{2}\right) = 0$$

$$-\lambda\left(\lambda^2 - \tfrac{1}{2}\right) + \tfrac{7}{24}\lambda + \tfrac{5}{24} = 0$$

$$-\lambda^3 + \tfrac{19}{24}\lambda + \tfrac{5}{24} = 0$$

$$24\lambda^3 - 19\lambda - 5 = 0$$

$\lambda = 1.$ is a solution

$\Rightarrow \lambda - 1 = 0$

$$\begin{array}{r|l}
 & 24\lambda^2 + 24\lambda + 5 \\ \hline
(\lambda - 1) & 24\lambda^3 - 19\lambda - 5 \\
 & (24\lambda^3 - 24\lambda^2) \\ \hline
 & 0 \quad 24\lambda^2 - 19\lambda \\
 & -(24\lambda^2 - 24\lambda) \\ \hline
 & 0 \quad 5\lambda - 5 \\
 & 5\lambda - 5 \\ \hline
 & 0 \quad 0
\end{array}$$

$$(\lambda-1)(24\lambda^2+24\lambda+5)=0$$

$\lambda_1=1$, $\quad 24\lambda^2+24\lambda+5=0$

$$\Rightarrow \lambda_{2,3} = \frac{-12}{24} \pm \frac{\sqrt{23}}{24}$$

Find the eigenvector corresponding to eigenvalue $\lambda_1=1$.

$$(P-\lambda_1 I)\vec{x}=0$$

$$\begin{pmatrix} -1 & 1/3 & 1/4 \\ 1/2 & -1 & 3/4 \\ 1/2 & 2/3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1/3 & 1/4 & 0 \\ 1/2 & -1 & 3/4 & 0 \\ 1/2 & 2/3 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1/3 & 1/4 & 0 \\ 0 & -5/3 & 7/4 & 0 \\ 0 & 5/3 & -7/4 & 0 \end{array}\right] \begin{array}{l} R_2 = 2R_2 + R_1 \\ R_3 = \\ 2R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1/3 & 1/4 & 0 \\ 0 & -5/3 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\tfrac{5}{3}x_2 + \tfrac{7}{4}x_3 = 0 \Rightarrow x_2 = \tfrac{21}{20}x_3 .$$

$$-x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 = 0$$

$$x_1 = \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 = \tfrac{21}{60}x_3 + \tfrac{1}{4}x_3 = \tfrac{12}{20}x_3$$

take $x_3 = 20$, $x_1 = 12$, $x_2 = 21$

$$\vec{X} = \begin{pmatrix} 12 \\ 21 \\ 20 \end{pmatrix}$$

multiply by $\frac{1}{53}$ to get a probability vector

$$\vec{X} = \begin{pmatrix} 12/53 \\ 21/53 \\ 20/53 \end{pmatrix} \quad (*)$$

Since the matrix P has only one eigenvalue $\lambda = 1$, $\vec{X}$ as given in $(*)$ is the equilibrium probability of the network.