Last day

* Complex number

$z = x + iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $i = \sqrt{-1}$

$x = Re(z)$, $y = Im(z)$

- $\mathbb{C} = \{ z = x + iy \mid x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1} \}$
  is called the complex plane.

- we can perform arithmetic operations such as
  - addition and subtraction
  - multiplication
  - division

- Every real number is a complex, $\Rightarrow \mathbb{R} \subset \mathbb{C}$

- Complex matrix

$$A = \begin{pmatrix} 4i & 2+3i \\ 4+i & 5 \end{pmatrix}$$

- complex linear system
  - homogeneous

NON-HOMOGENEOUS COMPLEX LINEAR SYSTEMS

Example: Find the solution of the linear system

$$2i\,x_1 + 2x_2 = 2i$$

$$3x_1 + (1-2i)x_2 = i$$

$\frac{2}{2i} = -i$

Solution

$$\left(\begin{array}{cc|c} 2i & 2 & 2i \\ 3 & (1-2i) & i \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -i & 1 \\ 3 & (1-2i) & i \end{array}\right) \frac{R_1}{2i}$$

$$\sim \left(\begin{array}{cc|c} 1 & -i & 1 \\ 0 & 1+i & -3+i \end{array}\right) \quad R_2 = R_2 - 3R_1$$

$$-(-3i)$$
$$(1-2i)+3i$$
$$1+i$$
$$i-3$$

From row 2, $(1+i)x_2 = -3+i$

$$x_2 = \frac{-3+i}{(1+i)} = \frac{-2+4i}{2} = -1+2i$$

From row 1, $x_1 - ix_2 = 1$

$$x_1 = 1 + ix_2 = 1 + i\left(\frac{-2+4i}{2}\right)$$

$$= 1 + i(-1+2i) = 1 - i - 2$$

$$= -1 - i$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1-i \\ -1+2i \end{pmatrix}$$

Inverse of a complex matrix

Given an $n \times n$ complex matrix A, we can find the inverse using the same idea we used for real matrices, that is construct the super-augmented matrix

$$[A|I]$$

reduce using elementary row operations to

$$[I|B]$$

and then $A^{-1} = B$.

COMPLEX EXPONENTIAL

Consider the differential equation (DE) (Initial Value Problem)

$$\frac{dy}{dt} = \alpha y(t) \quad , \quad y(0) = 1$$

We want to find a function $y(t)$ that satisfies this problem.

Let $y(t) = e^{\alpha t}$, $\frac{dy}{dt} = \alpha e^{\alpha t} = \alpha y(t)$

$$y(0) = e^0 = 1$$

$\therefore$ $y(t) = e^{\alpha t}$ is a solution of the problem.

Suppose $\alpha = i$, then the problem becomes

$$\frac{dy}{dt} = i y(t) \quad , \quad y(0) = 1$$

and the solution is $y = e^{it}$ ——— 1

Now,

Let $y = \cos(t) + i\sin(t)$, $y(0) = \cos(0) + i\sin(0) = 1$

$$\frac{dy}{dt} = -\sin(t) + i\cos(t) = i\cos(t) - \sin(t)$$

$$\frac{dy}{dt} = i\left(\cos(t) + i\sin(t)\right) = i\,y(t)$$

Since $y = \cos(t) + i\sin(t)$ ——— 2 satisfy the equation and the initial condition, it is also a solution of the problem.

Since we are solving an initial value problem, the solution of the problem is unique. Therefore, we can equate 1 and 2 to get

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

This is the complex exponential.

Some properties of complex exponents

(i) $e^{i\alpha + i\beta} = e^{i\alpha} \cdot e^{i\beta}$

(ii) Let $z = x + iy$

$$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$$

$$= e^{x}\left(\cos(y) + i\sin(y)\right)$$

$$e^{z} = e^{x}\cos(y) + i e^{x}\sin(y)$$

$$|e^{i\alpha}| = 1$$

$$|z| = \sqrt{x^2+y^2}$$

(iii) $$|e^{i\alpha}| = |\cos(\alpha) + i\sin(\alpha)|$$

$$= \sqrt{\cos^2(\alpha) + \sin^2(\alpha)}$$

$$= \sqrt{1} = 1$$

$$\Rightarrow |e^{i\alpha}| = 1$$

$x+iy$

(x, y)

$\sqrt{x^2+y^2}$

$(\cos(\alpha), \sin(\alpha))$

(iv) Let $z = x+iy$, $|e^z| = ?$

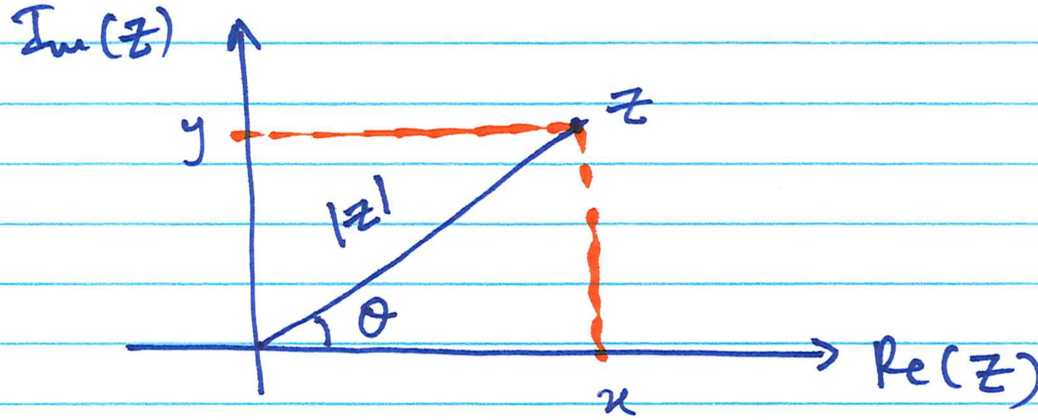
$$|e^z| = |e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x||e^{iy}|$$

$$= |e^x| \quad (\text{since } |e^{iy}| = 1)$$

$$|e^z| = e^x$$

Polar representation of a complex number

Let $z = x + iy$



$$\sin(\theta) = \frac{y}{|z|} \Rightarrow y = |z| \sin(\theta)$$

$$\cos(\theta) = x/|z| \Rightarrow x = |z| \cos(\theta)$$

$$z = x + iy = |z| \cos(\theta) + i|z| \sin(\theta)$$
$$= |z| \left(\cos(\theta) + i \sin(\theta) \right)$$

$$\boxed{z = |z| e^{i\theta}}$$

This is the polar representation of z.

θ = argument of z (radians)

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

(i) $z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$

$= r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

(ii) $\dfrac{1}{z_1} = \dfrac{1}{r_1 e^{i\theta_1}} = \dfrac{1}{r_1} \cdot \dfrac{1}{e^{i\theta_1}} = \dfrac{1}{r_1} e^{-i\theta_1}$

$z = x+iy$ with $|z| = 1$

$\dfrac{1}{z} = \dfrac{1}{|z| e^{i\theta}} = \dfrac{1}{1} \cdot \dfrac{1}{e^{i\theta}} = e^{-i\theta}$

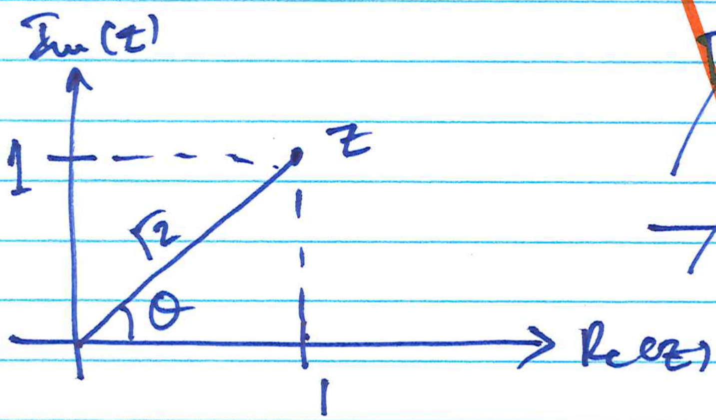
$\bar{z} = e^{-i\theta}$, $\bar{z} = x - iy$

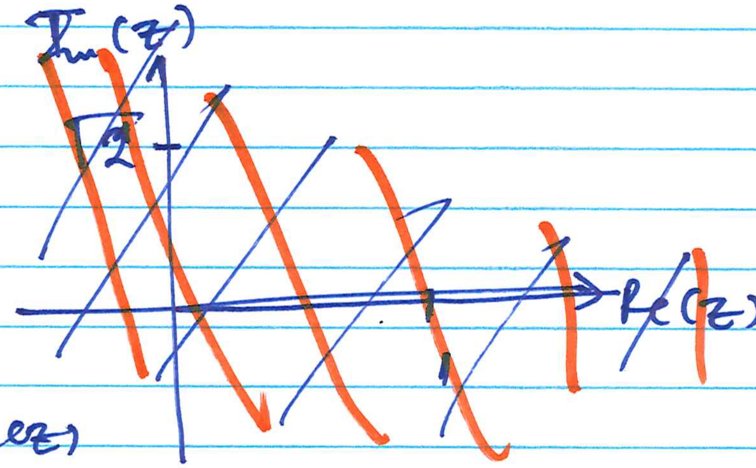
If z is a complex number with $|z| = 1$,

then $\dfrac{1}{z} = \bar{z}$

(iii) $\dfrac{z_1}{z_2} = \dfrac{(r_1 e^{i\theta_1})}{(r_2 e^{i\theta_2})} = \dfrac{r_1}{r_2}\,\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = r_1/r_2\, e^{i(\theta_1-\theta_2)}$

Examples: Let $z = 1+i$, write z in polar coordinates.





$\left(\cos(\theta) = \dfrac{1}{\sqrt{2}}\right)$

To write z in polar coordinates,

$z = |z|\, e^{i\theta}$

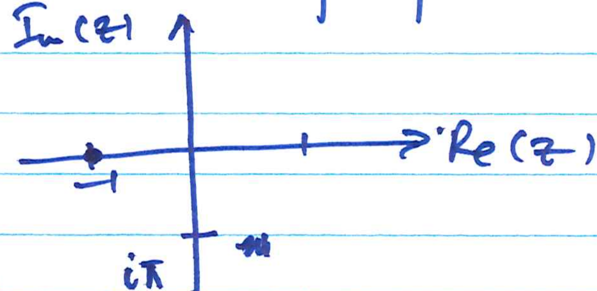
$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\cos(\theta) = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(1/\sqrt{2}\right) = \pi/4$

$\therefore\ z = \sqrt{2}\, e^{i\pi/4}$

Example: Find four roots of the following equation

$$z^4 + 1 = 0$$



$$\Rightarrow z^4 = -1$$

$$-1 = e^{i\pi} \quad \text{--- (2)}$$

Let $z = r e^{i\theta}$

$$z^4 = r^4 e^{i4\theta} \quad \text{--- (1)}$$

equate (1) and (2)

α

$+ 2\pi k$

$\cancel{k = 0, 1,}$

$k = 0, 1, 2, \ldots$

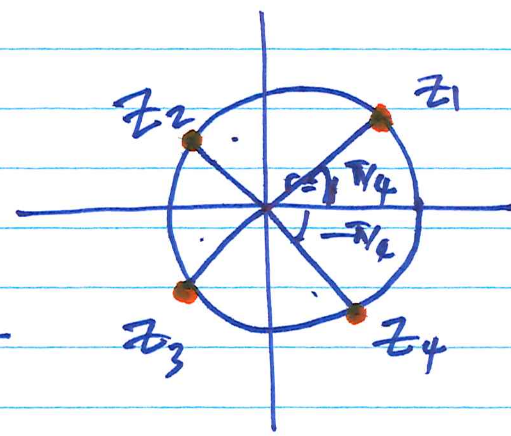
$$z^4 = -1$$

$$\Rightarrow r^4 e^{i4\theta} = e^{i\pi}$$

$$\Rightarrow r^4 = 1 \Rightarrow r = 1$$

$$4\theta = \pi, 3\pi, 5\pi, 7\pi$$

$$\theta = \pi/4, \ 3\pi/4, \ \frac{5\pi}{4}, \ 7\pi/4$$



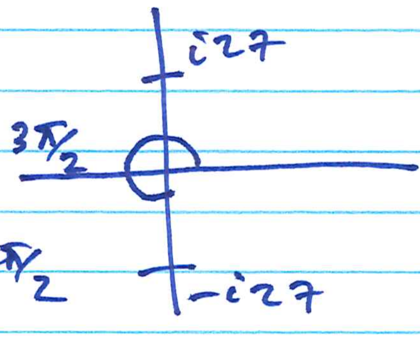
The roots of the equation are

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{i3\pi/4}, \quad z_3 = e^{i5\pi/4}$$

$$z_4 = e^{i7\pi/4}$$

Example: Find 3 roots of the equation

$$z^3 + i27 = 0$$



$$z^3 = -i27$$

Let $z = re^{i\theta}$, $-i27 = 27e^{i3\pi/2}$

$\Rightarrow z^3 = -27i$ $\qquad \alpha + 2\pi K$

$\Rightarrow r^3 e^{i3\theta} = 27e^{i3\pi/2}$

$\Rightarrow r^3 = 27 \Rightarrow r = 3$

$\Rightarrow 3\theta = 3\pi/2, 7\pi/2, 11\pi/2$

$\Rightarrow \theta = \pi/2, 7\pi/6, 11\pi/6$

$z_1 = 3e^{i\pi/2}$, $z_2 = 3e^{i7\pi/6}$

$z_3 = 3e^{i11\pi/6}$