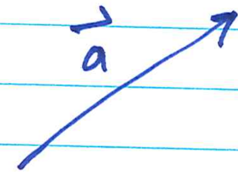
Last day

$\vec{a}$ in 2D



$\vec{a} = (a_1, a_2)$

where a_1 is the component of $\vec{a}$ in x-direction
and a_2 is the component of $\vec{a}$ ✓ y-direction

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

$$\vec{a} = (a_1, a_2, \dots, a_n)$$

The length of $\vec{a}$ is

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

Dot products

$\vec{a}$ and $\vec{b}$ $\qquad \vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

and dot product is a scalar.

We say two vectors are orthogonal if the dot product of the two vectors is zero.

Example

Let $\vec{a} = (1, 3)$ and $\vec{b} = (m, 2)$
Find the value of m for which vector $\vec{a}$ is orthogonal to vector $\vec{b}$

Solution

a If $\vec{a}$ and $\vec{b}$ are orthogonal, then
$\vec{a} \cdot \vec{b} = 0$

Let us compute $\vec{a} \cdot \vec{b}$

$$\vec{a} \cdot \vec{b} = (1, 3) \cdot (m, 2)$$
$$= m + 6$$

$$\vec{a} \cdot \vec{b} = 0$$
$$\Rightarrow \quad m + 6 = 0$$
$$m = -6$$

Example:

Suppose $\vec{a} = (-1, 2, 3)$ and $\vec{b} = (4, 3, m)$

Find the value of m for which $\vec{a}$ is perpendicular to $\vec{b}$.

Solution

$\vec{a}\cdot\vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are perpendicular

$$\vec{a}\cdot\vec{b} = (-1, 2, 3)\cdot(4, 3, m)$$

$$= -4 + 2(3) + 3(m)$$

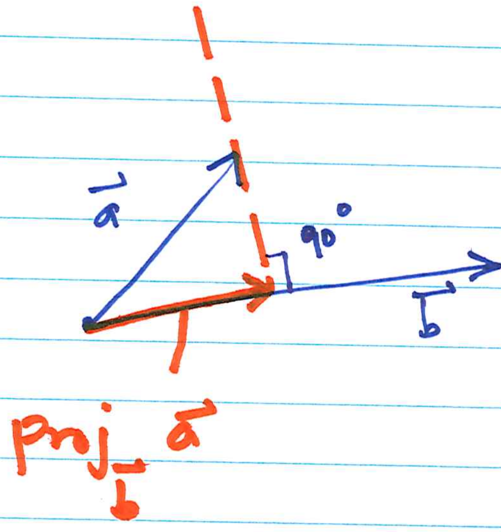
$$= -4 + 6 + 3m = 2 + 3m$$

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \quad 2 + 3m = 0$

$$m = \frac{-2}{3}$$

Projection



Projection of vector $\vec{a}$ onto $\vec{b}$ is

$$\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

This is also called the component of $\vec{a}$ in the direction $\vec{b}$.

$$\text{Proj}_{\vec{b}}\,\vec{a} = \left(\text{comp}_{\vec{b}}\,\vec{a}\right)\cdot\frac{\vec{b}}{\|\vec{b}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\cdot\vec{b}$$

Example:

$\vec{a} = (2, 3, 1)$ and $\vec{b} = (1, 4, 3)$

Find the projection vector $\vec{a}$ in the direction of $\vec{b}$.

Solution:

$$\text{proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$$

$$\vec{a} \cdot \vec{b} = (2, 3, 1) \cdot (1, 4, 3)$$
$$= 2 + 12 + 3 = 17$$

$$\|\vec{b}\| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{1 + 16 + 9} = \sqrt{26}$$

$$\|\vec{b}\|^2 = 26$$

$$\text{proj}_{\vec{b}} \vec{a} = \frac{17}{26} \cdot (1, 4, 3) = \left(\frac{17}{26}, \frac{68}{26}, \frac{51}{26} \right)$$

$$\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \frac{17}{\sqrt{26}}$$

Example

$\vec{a} = (1, 4, 0)$ and $\vec{b} = (2, -1, 5)$

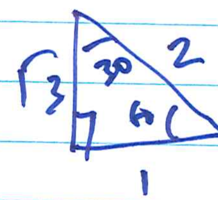
(a) Compute the angle between $\vec{a}$ and $\vec{b}$

Suppose θ is angle between $\vec{a}$ ~~b~~ and $\vec{b}$

Then

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$$

$$\vec{a} \cdot \vec{b} = (1, 4, 0) \cdot (2, -1, 5)$$
$$= 2 + (-4) + 0$$
$$= 2 - 4 = -2$$



$$\|\vec{a}\| = \sqrt{1^2 + 4^2 + 0^2} = \sqrt{1 + 16} = \sqrt{17}$$

$$\|\vec{b}\| = \sqrt{2^2 + (-1)^2 + 5^2} = \sqrt{4 + 1 + 25} = \sqrt{30}$$

$$\cos(\theta) = \frac{-2}{(\sqrt{17})(\sqrt{30})} = \frac{-2}{\sqrt{510}} = -0.08856$$

$$\theta = \arccos(-0.08856) = \underline{\underline{95.0809^\circ}}$$

Unit vectors

A unit vector is a vector with length 1.
Let $\vec{a}$ be a vector.
We want to find another vector in the direction of $\vec{a}$ such that this vector has length 1.

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

Example!
Let $\vec{a} = (1, 4)$. Find a unit vector in the direction of vector $\vec{a}$.

Solution
This means that we want to compute the vector

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

$$\|\vec{a}\| = \sqrt{1^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$$

$$\underline{\underline{\hat{a}}} = \frac{1}{\sqrt{17}}(1, 4) = \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$$

$$\|\hat{a}\| = \sqrt{\left(\frac{1}{\sqrt{17}}\right)^2 + \left(\frac{4}{\sqrt{17}}\right)^2} = \sqrt{\frac{1}{17} + \frac{16}{17}} = \sqrt{\frac{17}{17}} = \sqrt{1} = 1$$

(1) If $\vec{a} = (1, 3, 5)$. Find ~~the~~ a unit vector in the direction of $\vec{a}$.

Solution

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

$$\|\vec{a}\| = \sqrt{1^2+3^2+5^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\hat{a} = \frac{1}{\sqrt{35}}(1, 3, 5) = \left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right)$$

$$\|\hat{a}\| = \sqrt{\left(\frac{1}{\sqrt{35}}\right)^2 + \left(\frac{3}{\sqrt{35}}\right)^2 + \left(\frac{5}{\sqrt{35}}\right)^2}$$

$$= \sqrt{\frac{1}{35} + \frac{9}{35} + \frac{25}{35}} = \sqrt{\frac{1+9+25}{35}} = \sqrt{\frac{35}{35}}$$

$$\|\hat{a}\| = \underline{\underline{1}}$$

Determinants of Matrices

In 2D, Let A be a 2×2 matrix,

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

The determinant of A is defined as

$$\det(A) = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= a_1 b_2 - a_2 b_1$$

Example:

(i) Let $A = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$. Find the determinant of A.

Solution

$$\det(A) = \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix} = 4(2) - 3(5) = 8 - 15 = \underline{\underline{-7}}$$

(ii) compute $\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$

$$\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = 2(5) - 1(4) = 10 - 4 = \underline{\underline{6}}$$

In 3 dimension,

Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\det(A) = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - c_1 b_3) + a_3 (b_1 c_2 - c_1 b_2)$$

Example:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{pmatrix}$$. Find the determinant of A.

$$\det(A) = 1 \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} - 3 \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= 1 \big(1(2) - 3(5)\big) - 3\big(4(2) - 2(3)\big) + 2\big(4(5) - 2(1)\big)$$

$$= 1 (2 - 15) - 3 (8 - 6) + 2 (20 - 2)$$

$$= -13 - 6 + 36 = \underline{\underline{17}}$$

Example:

Compute
$$\begin{vmatrix} 6 & 1 & 2 \\ 3 & -1 & 4 \\ 2 & 8 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 1 & 2 \\ 3 & -1 & 4 \\ 2 & 8 & 7 \end{vmatrix} = 6 \begin{vmatrix} -1 & 4 \\ 8 & 7 \end{vmatrix} - 1 \begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ 2 & 8 \end{vmatrix}$$

$$= 6(-39) - 1(13) + 2(26)$$

$$= -234 - 13 + 52 = \underline{\underline{-195}}$$

Note

We can only compute the determinant for square matrices.

Application of determinants

Let A be a 2×2 matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

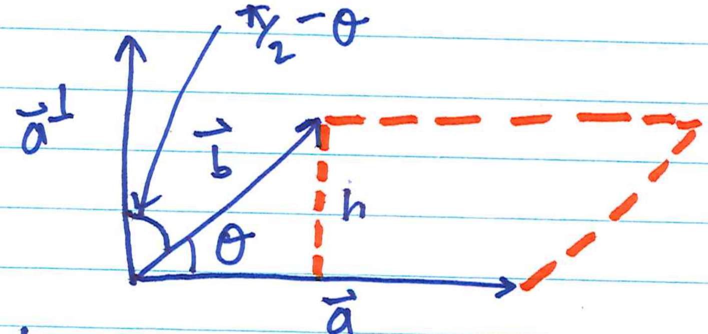
Let the first row of A be $\vec{a} = (a_1, a_2)$ and the second row be vector $\vec{b} = (b_1, b_2)$

Define a vector that is orthogonal to $\vec{a}$

$$\vec{a}^{\perp} = (-a_2, a_1)$$

Check

$$\vec{a}\cdot\vec{a}^{\perp} = (a_1, a_2)\cdot(-a_2, a_1)$$
$$= a_1(-a_2) + a_2 a_1$$
$$= -a_1 a_2 + a_1 a_2 = 0$$



$$\sin(\theta) = \frac{h}{\|\vec{b}\|} \Rightarrow \underline{\underline{h = \|\vec{b}\| \sin(\theta)}}$$

$$\vec{a}^{\perp} \cdot \vec{b} = (-a_2, a_1) \cdot (b_1, b_2)$$

$$= -b_1 a_2 + a_1 b_2$$

$$= a_1 b_2 - a_2 b_1$$

$$\vec{a}^{\perp} \cdot \vec{b} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \det(A) \qquad (1)$$

The angle between $\vec{a}^{\perp}$ and $\vec{b}$ is $\pi/2 - \theta$

$$\cos(\pi/2 - \theta) = \frac{\vec{a}^{\perp} \cdot \vec{b}}{\|\vec{a}^{\perp}\| \|\vec{b}\|} = \frac{\vec{a}^{\perp} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

$$\vec{a}^{\perp} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\pi/2 - \theta)$$

from (1),

$$\vec{a}^{\perp} \cdot \vec{b} = \det(A) = \|\vec{a}\| \|\vec{b}\| \cos(\pi/2 - \theta)$$

$$\det(A) = \|\vec{a}\| \|\vec{b}\| \sin(\theta) \qquad (2)$$

The area of a parallelogram is given by

Area = base × height

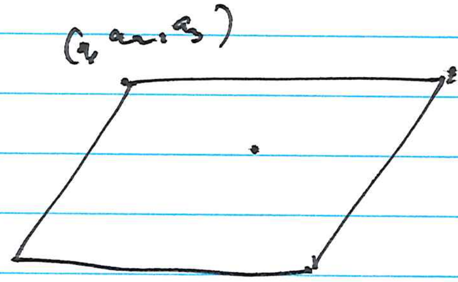
base = $\|\vec{a}\|$

height = $\|\vec{b}\| \sin(\theta)$

Area = $\|\vec{a}\| \, \|\vec{b}\| \sin(\theta)$

from (2), we see that the determinant of matrix A is the area of the parallelogram whose sides are the rows of A.

ie Area of parallelogram = $|\det(A)|$.



Example

Find the ~~[illegible]~~ area of the parallelogram spanned by $\vec{a} = (1,1)$ and $\vec{b} = (1,4)$

Solution

~~Con~~ Construct a matrix A

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

The area of the parallelogram is $\|\det(A)\|$

$$\det(A) = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 4 - 1 = 3$$

$$\|\det(A)\| = \underline{\underline{3}}$$