Last day

* Matrix operations

  * Column vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$

  * row vector $(a_1, a_2, \ldots, a_m)$

$$(a_1, a_2, \ldots, a_m)^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

* matrix addition
* Scalar multiplication

* today : matrix multiplication.

* Matrix multiplication

Let A be an $m \times p$ matrix and B be an $p \times n$ matrix

$$A = \begin{pmatrix} \text{---} & A_1 & \text{---} \\ \text{---} & A_2 & \text{---} \\ & \vdots & \\ \text{---} & A_m & \text{---} \end{pmatrix}, \quad B = \begin{pmatrix} | & | & & | \\ B_1 & B_2 & \cdots & B_n \\ | & | & & | \end{pmatrix}$$

$$A \times B = \begin{pmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & \cdots & A_1 \cdot B_n \\ A_2 \cdot B_1 & A_2 \cdot B_2 & \cdots & A_2 \cdot B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_m \cdot B_1 & A_m \cdot B_2 & \cdots & A_m \cdot B_n \end{pmatrix}$$

NB:

The number of columns of A must equal the number of rows of B.

Example:

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & \cancel{4}\,5 \\ 4 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$\dim(A) = 3 \times 4$

$\dim(B) = 4 \times 3$

$$A \times B = \begin{pmatrix} 2+6+16+5 & 2+9+8+5 & 2+15+8+5 \\ 2+2+16+1 & 2+3+8+1 & 2+5+8+1 \\ 1+4+12+4 & 1+6+6+4 & 1+10+6+4 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 29 & 24 & 30 \\ 21 & 14 & 16 \\ 21 & 17 & 21 \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 3 & 4 & 2 & 5 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 4 \\ 3 & 2 & 3 & 5 \end{pmatrix}$$

$A \times B$?

We can not multiply the two matrices because the number of columns of A does not equal the number of rows of B.

Properties of matrix multiplication

Let A, B, and C be matrices and α and β be a scalar

1. $A+B = B+A$
2. $A+(B+C) = (A+B)+C$
3. $\alpha(A+B) = \alpha A + \alpha B$ Also
4. $(\alpha+\beta)A = \alpha A + \beta A$
5. $(\alpha\beta)A = \alpha(\beta A)$
6. $IA = A$ (I is the identity matrix.)
7. $A+0 = A$ (0 is the zero matrix)
8. $A - A = A + (-1)A = 0$
9. $A(B+C) = AB + AC$
10. $A(BC) = (AB)C$
11. $(A+B)C = AC + BC$
12. $\alpha(AB) = (\alpha A)B = A(\alpha B)$

Is $AB = BA$? NO

$$AB \neq BA.$$

LINEAR TRANSFORMATIONS AND MATRICES.

Def:

Let T be a transformation. T is linear if for any vectors $\vec{X}$ and $\vec{Y}$, and scalar α, β

$$T(\alpha\vec{X} + \beta\vec{Y}) = \alpha T(\vec{X}) + \beta T(\vec{Y})$$

This property can also be written as

(i) $T(\vec{X}+\vec{Y}) = T(\vec{X}) + T(\cancel{\alpha}\vec{Y})$

(ii) $T(\alpha\vec{X}) = \alpha T(\vec{X})$.

Def: A transformation is a function whose inputs are vectors: E.g. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

defined by $T(\vec{X}) = (2x_1, 2x_2) = 2\vec{X}$

for $\vec{X} = (x_1, x_2) \in \mathbb{R}^2$

Example: Determine if the transformation is

$$T(\vec{x}) = (2x_1 + 3x_2,\ x_1 + x_2),\ \vec{x} \in \mathbb{R}^2.$$

We need to check if

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$

and $T(\alpha \vec{x}) = \alpha T(\vec{x})$.

Let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$

$$\vec{x} + \vec{y} = (x_1 + y_1,\ x_2 + y_2)$$

$$T(\vec{x} + \vec{y}) = T\big((x_1 + y_1,\ x_2 + y_2)\big) =$$

$$= \big[2(x_1 + y_1) + 3(x_2 + y_2),\ x_1 + y_1 + x_2 + y_2\big]$$

$$= \big[2x_1 + 2y_1 + 3x_2 + 3y_2,\ x_1 + y_1 + x_2 + y_2\big]$$

$$= \big[(2x_1 + 3x_2) + (2y_1 + 3y_2),\ (x_1 + x_2) + (y_1 + y_2)\big]$$

$$= (2x_1 + 3x_2,\ x_1 + x_2) + (2y_1 + 3y_2,\ y_1 + y_2)$$

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$

✓

Let α be a scalar

$$T(\alpha\vec{x}) = T\left((\alpha x_1, \alpha x_2)\right)$$

$$= \left[2\alpha x_1 + 3\alpha x_2 , \alpha x_1 + \alpha x_2\right]$$

$$= \left[\alpha(2x_1 + 3x_2) , \alpha(x_1 + x_2)\right]$$

$$= \alpha\left(2x_1 + 3x_2 , x_1 + x_2\right)$$

$$T(\alpha\vec{x}) = \alpha T(\vec{x}) \checkmark$$

Since the two properties are satisfied,

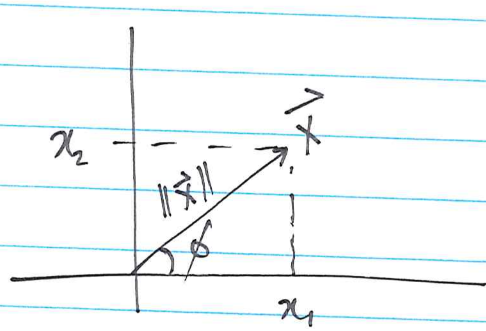
T is a linear transformation.

<u>Exercise</u>: Is the transformation linear?

$$T(\vec{x}) = \left(x_1 + 3x_2 , x_1 x_2\right) , \vec{x} \in \mathbb{R}^2$$

ROTATIONS IN TWO DIMENSIONS

Let $\vec{x} = (x_1, x_2)$ be a vector in 2D and let ϕ be the angle the vector makes with the x-axis.

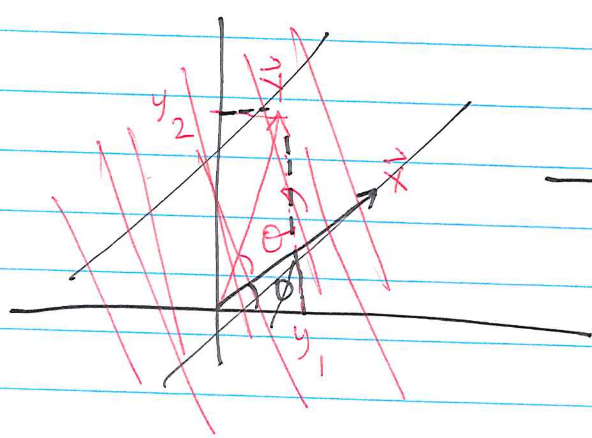


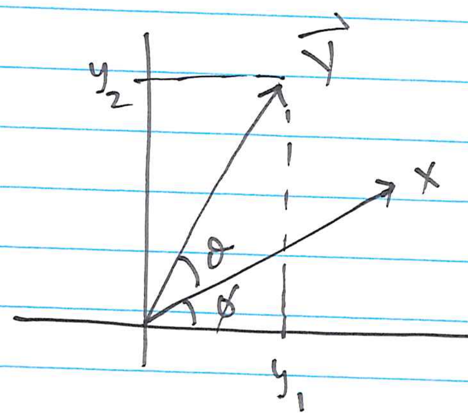
SOH CAH TOA

$$\left.\begin{aligned} \sin\phi = \frac{x_2}{\|\vec{x}\|} &\Rightarrow x_2 = \|\vec{x}\| \sin\phi . \\ \cos\phi = \frac{x_1}{\|\vec{x}\|} &\Rightarrow x_1 = \|\vec{x}\| \cos\phi . \end{aligned}\right\} \quad (1)$$

Let $\vec{Y}$ be the rotation of the vector $\vec{X}$ in counterclockwise direction by an angle θ.

$$\vec{Y} = Rot_\theta \vec{X} = (y_1, y_2)$$





$$y_1 = \|\vec{x}\| \cancel{\sin} \cos(\phi + \theta)$$
$$y_2 = \|\vec{x}\| \sin(\phi + \theta).$$

Recall,

$$\sin(\phi + \theta) = \sin\phi \cos\theta + \cos\phi \sin\theta$$
$$\cos(\phi + \theta) = \cos\phi \cos\theta - \sin\phi \sin\theta$$

$$\therefore \quad y_1 = \|\vec{x}\| \left(\cos\phi \cos\theta - \sin\phi \sin\theta \right)$$
$$= \left(\|\vec{x}\| \cos\phi \right) \cos\theta - \left(\|\vec{x}\| \sin\phi \right) \sin\theta$$
$$y_1 = x_1 \cos\theta - x_2 \sin\theta. \qquad \text{(using (1))}$$

$$y_2 = \|\vec{x}\| \left(\sin\phi \cos\theta + \cos\phi \sin\theta \right)$$
$$= \left(\|\vec{x}\| \sin\phi \right) \cos\theta + \left(\|\vec{x}\| \cos\phi \right) \sin\theta$$
$$y_2 = x_2 \cos\theta + x_1 \sin\theta.$$

$$\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_2 \cos\theta + x_1 \sin\theta \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}$$

$$\vec{Y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Therefore, the rotation matrix is

$$\text{Rot}_\theta \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for counterclockwise rotation.

Que: suppose we want to rotate $\vec{x}$ by angle θ in clockwise direction, what is the rotation matrix?

θ in clockwise direction is $-\theta$ in counterclockwise direction.

$$\therefore \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Example: Find the vector obtained by rotating a vector $\vec{X} = (3, 4)$ by an angle of $45°$ in counterclockwise direction.

$\theta = 45°$

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$Rot_\theta \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

$$Rot_\theta \vec{X} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ \frac{7\sqrt{2}}{2} \end{pmatrix}$$

PROJECTIONS IN 2D

Recall, the projection of a vector $\vec{x}$ in the direction of another vector $\vec{a}$ is given by

$$\text{Proj}_{\vec{a}}\,\vec{x} = \frac{(\vec{x}\cdot\vec{a})}{\|\vec{a}\|^2}\,\vec{a}$$

Projection is a linear transformation (exercise!)

Suppose the vector $\vec{a}$ has ~~unit~~ unit length i.e. $\|\vec{a}\| = 1$

then $\text{proj}_{\vec{a}}\,\vec{x} = (\vec{x}\cdot\vec{a})\,\vec{a}$

if $\vec{x} = (x_1, x_2)$ and $\vec{a} = (a_1, a_2)$

$$\text{Proj}_{\vec{a}}\,\vec{x} = \left[(x_1, x_2)\cdot(a_1, a_2)\right](a_1, a_2)$$

$$= \left[a_1x_1 + a_2x_2\right](a_1, a_2)$$

$$= \left(a_1(a_1x_1 + a_2x_2),\; a_2(a_1x_1 + a_2x_2)\right)$$

Let us write the vector as a column vector

$$\text{proj}_{\vec{a}}\,\vec{x} = \begin{pmatrix} a_1(a_1x_1 + a_2x_2) \\ a_2(a_1x_1 + a_2x_2) \end{pmatrix} = \begin{pmatrix} a_1^2x_1 + a_1a_2x_2 \\ a_1a_2x_1 + a_2^2x_2 \end{pmatrix}$$

$$\mathrm{Proj}_{\vec{a}}\,\vec{x} = \underbrace{\begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix}}_{\text{Projection matrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Suppose $\vec{a}$ is not a unit vector.

$\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|}$ is a ~~unit~~ unit vector in the same as $\vec{a}$.

$= \left(\dfrac{a_1}{\|\vec{a}\|}, \dfrac{a_2}{\|\vec{a}\|} \right)$

$$\mathrm{Proj}_{\vec{a}}\,\vec{x} = \begin{pmatrix} \dfrac{a_1^2}{\|\vec{a}\|^2} & \dfrac{a_1 a_2}{\|\vec{a}\|^2} \\ \dfrac{a_1 a_2}{\|\vec{a}\|^2} & \dfrac{a_2^2}{\|\vec{a}\|^2} \end{pmatrix} = \frac{1}{\|\vec{a}\|^2} \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix} \quad (*)$$

We have

$\vec{a} = (a_1, a_2)$. Let θ be the angle $\vec{a}$ makes with the x-axis.

Recall, $a_1 = \|\vec{a}\| \cos\theta$

$a_2 = \|\vec{a}\| \sin\theta$.

Substituting a_1 and a_2 into $(*)$ gives the projection matrix

$$\mathrm{Proj}_{\vec{a}}\,\vec{x} = \frac{1}{2} \begin{pmatrix} 1 + \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 1 - \cos(2\theta) \end{pmatrix}$$

Example: Find the projection of vector $\vec{x} = (2, 3)$ in ~~the~~ the direction of vector $\vec{a} = (-1, 4)$.

$$\operatorname{Proj}_{\vec{a}} = \frac{1}{\|\vec{a}\|^2} \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix} \quad , \quad \|\vec{a}\|^2 = (-1)^2 + (4)^2 = 17$$

$$\operatorname{Proj}_{\vec{a}} = \frac{1}{17} \begin{pmatrix} (-1)^2 & (-1)(4) \\ (-1)(4) & (4)^2 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix}$$

$$\operatorname{Proj}_{\vec{a}} \vec{x} = \frac{1}{17} \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =$$

$$= \frac{1}{17} \begin{pmatrix} -10 \\ 40 \end{pmatrix} = \begin{pmatrix} -10/17 \\ 40/17 \end{pmatrix}$$