Last day

* complex exponentials

Let $z = x + iy$,

$$e^{iz} = e^{i(x+iy)} = e^{ix - y} = e^{-y} e^{ix}$$

$$= e^{-y}\left(\cos(x) + i\sin(x)\right)$$

$$= e^{-y}\cos(x) + i e^{-y}\sin(x)$$

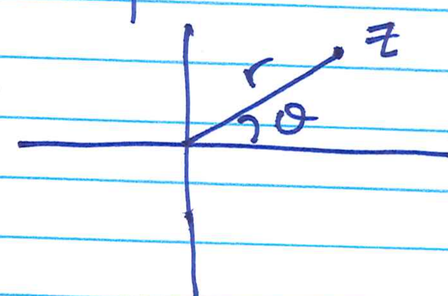
If $x = Re(z) = 0$, $z = iy$

$$e^{z} = e^{iy} = \cos(y) + i\sin(y).$$

* polar resentation of complex numbers

Let $z = x + iy$, then in polar coordinates

$$z = r e^{i\theta}$$



where $r = |z|$

θ (argument of z) is the angle z makes with the positive x-axis.

EIGENVALUES AND EIGENVECTORS

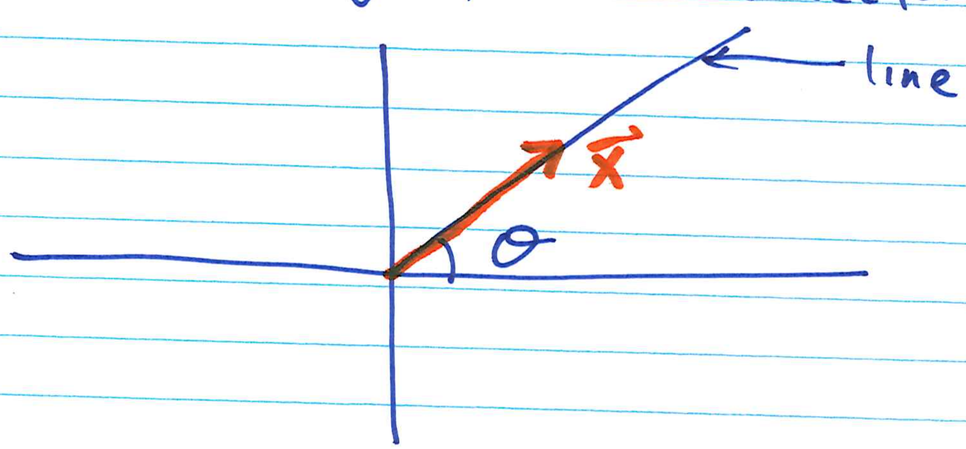
Let A be an $n \times n$ matrix. A number λ and a vector $\vec{x}$ is an eigenvalue eigenvector pair of A if;

(i) $\vec{x} \neq \vec{0}$

(ii) $A\vec{x} = \lambda \vec{x}$

If $\exists$ (there exist) such λ and a vector $\vec{x}$, then λ is ~~the~~ an eigenvalue of A with corresponding eigenvector $\vec{x}$.

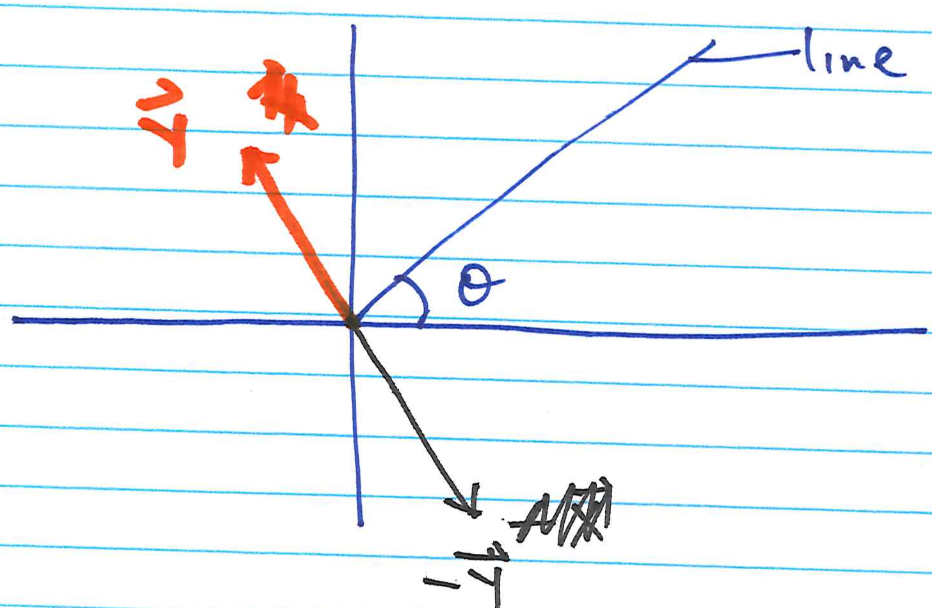
Consider the reflection matrix, R about a line that ~~ma~~ makes an angle of Θ with the x-axis, and let $\vec{x}$ be a vector on the line



$$R\vec{x} = \vec{x} \qquad \text{———— } (*)$$

relate this to $A\vec{x} = \lambda\vec{x}$.

This shows that $(*)$ is an eigenvalue problem where $\lambda = 1$ is an eigenvalue with corresponding eigenvector $\vec{x}$.



$$R\vec{y} = -\vec{y}$$

$\Rightarrow$ $\lambda = -1$ is an eigenvalue with eigenvector $\vec{y}$.

We observe that these are the only two vectors ~~that can give us a scalar multiple~~ which when multiplied by A will give us a scalar multiple of the original vector.

This implies that the reflection matrix has only two eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors $\vec{x}$ and $\vec{y}$, respectively.

These eigenvectors are linearly independent and form a basis for $\mathbb{R}^2$.

They also form a basis of eigenvectors of the reflection matrix.

Let A be an $n \times n$ matrix and λ be an eigenvalue of A with eigenvector $\vec{x}$. Then

$$A\vec{x} = \lambda \vec{x}$$

Let ~~α a sca~~ $\alpha \neq 0$ be a scalar.

$$A(\alpha\vec{x}) = A\alpha\vec{x} = \alpha A\vec{x} = \alpha\lambda\vec{x}$$

$$A(\alpha\vec{x}) = \lambda(\alpha\vec{x})$$

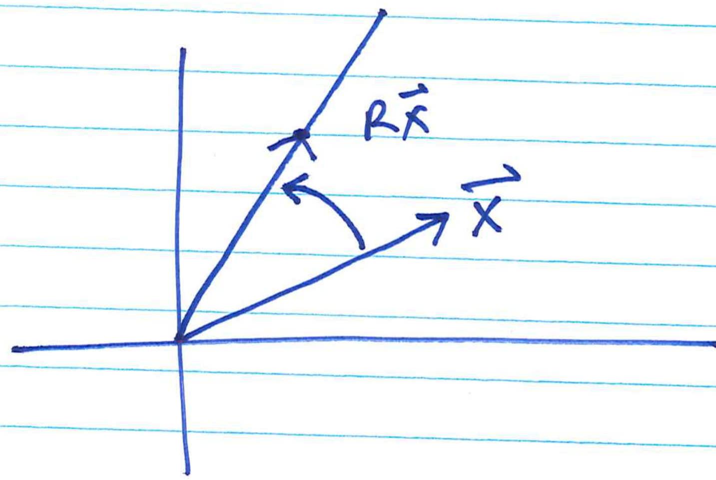
$\therefore$ $\alpha\vec{x}$ is also an eigenvector of eigenvalue λ.

This implies that the eigenvector of an ~~eigen~~ eigenvalue is not uniquely determine.

And this also shows that the direction of the eigenvector is the important factor.

Let us consider the rotation matrix. Let R be the matrix of rotation by an angle $45°$.

Let $\vec{x}$ be a vector



Observe that the direction of $\vec{x}$ is changed by R.

$\Rightarrow$ R does not have an eigenvalue.

Computing the eigenvalue and eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$.

Then

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda \vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

Let $B = A - \lambda I$, then

$$B\vec{x} = 0 \qquad \text{(homogeneous system)}$$

If $\vec{x} \neq 0$, then $\det(B) = 0$

$$\Rightarrow \quad \boxed{\det(A - \lambda I) = 0} \qquad I \text{ is identity matrix.}$$

this is called the characteristic equation.

After we obtain the eigenvalue, we can use

$$(A - \lambda I)\vec{x} = \vec{0}$$

to get the eigenvectors.

Example: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

Solution:

Let λ be an eigenvalue of A.

then

$$\det(A - \lambda I) = 0$$

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 9 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\lambda_1 = 5 \quad \& \quad \lambda_2 = -1$$

$\therefore$ these are the eigenvalues of A.

Let us find the eigenvector.

For $\lambda_1 = 5$, $\quad (A - \lambda_1 I)\vec{x} = \vec{0}$

$$\left(\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 3 & -3 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \quad 3x_1 - 3x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_2 \text{ is free}$$

$$\Rightarrow \quad \vec{x} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (\alpha \neq 0, \text{ is constant})$$

$\lambda_2 = -1, \quad (A - \lambda_2 I)\vec{x} = \vec{0}$

$$\left(\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$3x_1 + 3x_2 = 0$$

$$x_1 = -x_2$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x} = \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad (\beta \neq 0, \text{ constant})$$

We have two eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$

with corresponding eigenvectors

$$\vec{x}_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{X}_2 = \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

respectively.

$\alpha \neq 0$ and $\beta \neq 0$ are scalars.

Example : (Repeated eigenvalues)

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}$$

Let λ be an eigenvalue of A, then

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix} = 0$$

$$-\lambda\left[-\lambda(5-\lambda)+6\right] + 4\left[-1(5-\lambda)+3\right] - 6\left[-2+\lambda\right] = 0$$

Simplifying, we get

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

(characteristic polynomial)

By ~~insp~~ the inspection, we get $\lambda = 1$ as a root of the polynomial. $\Rightarrow \lambda - 1 = 0$

$$
\begin{array}{r|l}
 & \lambda^2 - 4\lambda + 4 \\
\hline
\lambda - 1 & \lambda^3 - 5\lambda^2 + 8\lambda - 4 \\
 & -(\lambda^3 - \lambda^2) \\
\hline
 & 0 \quad -4\lambda^2 + 8\lambda \\
 & \quad -(-4\lambda^2 + 4\lambda) \\
\hline
 & \qquad\qquad 4\lambda - 4 \\
 & \qquad\qquad 4\lambda - 4 \\
\hline
 & \qquad\qquad 0 \quad\; 0
\end{array}
$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

$\therefore$ the eigenvalues are

$\lambda_1 = 1$ and $\lambda_2 = 2$

For $\lambda_1 = 1$, $(A - \lambda_1 I)\vec{x} = \vec{0}$

$$\Rightarrow \left[\begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from row 2, $x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

from row 1

$$x_1 + 2x_2 + 4x_3 = 0$$

$$x_1 = -2x_2 - 4x_3 = -2(-x_3) - 4x_3 = 2x_3 - 4x_3$$

$$x_1 = -2x_3$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

x_3 is free.

For $\lambda_2 = 2$, $(A - \lambda_2 I)\vec{X} = \vec{0}$

$$\Rightarrow \left(\begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \begin{array}{l} R_1/-2 \\ -1R_2 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = -2x_2 - 3x_3$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

x_2 and x_3 are free.

x_2 and x_3 can not both be zero.

$\therefore$ the eigenvector of $\lambda_2 = 2$ is

$$\vec{X} = \alpha_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

where α_1 and α_2 are scalars and cannot both be zero.

For example: take $\alpha_1 = 0$, and $\alpha_2 = 1$,

$\vec{X} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of λ_2

OR you can take $\alpha_1 = 1$ and $\alpha_2 = 0$

$\vec{X} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is also an eigenvector of λ_2.