Last day

* triple product

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$

the triple product of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

* volume of a parallelepiped is $|\vec{a}\cdot(\vec{b}\times\vec{c})|$

* lines in 2D
  * parametric form of lines in 2D.
    * line passes through the origin and in the direction of vector $\vec{a}$

$$\vec{x} = t\vec{a} \quad , \text{ for some value of } t$$

    * line passes through the point $\vec{q}$, and in the direction of vector $\vec{a}$

$$\vec{x} = \vec{q} + t\vec{a} \quad , \quad t \in \mathbb{R}$$

$\mathbb{R}$

Que! what is $\vec{q}$ if the line passes through the origin?

We know that if ~~the~~ a line passes through the origin in the direction of $\vec{a}$, then we have

$$\vec{x}_1 = t\vec{a}$$

If it passes through $\vec{q}$ in the same direction,

$$\vec{x}_2 = \vec{q} + t\vec{a}$$

We want to find $\vec{q}$ such that $\vec{x}_1 = \vec{x}_2$

$$\vec{x}_1 = \vec{x}_2$$

$$\Rightarrow \quad t\vec{a} = \vec{q} + t\vec{a}$$

$\Rightarrow$ $\vec{q}$ must be $(0, 0) = \vec{0}$

<u>NOTE</u>

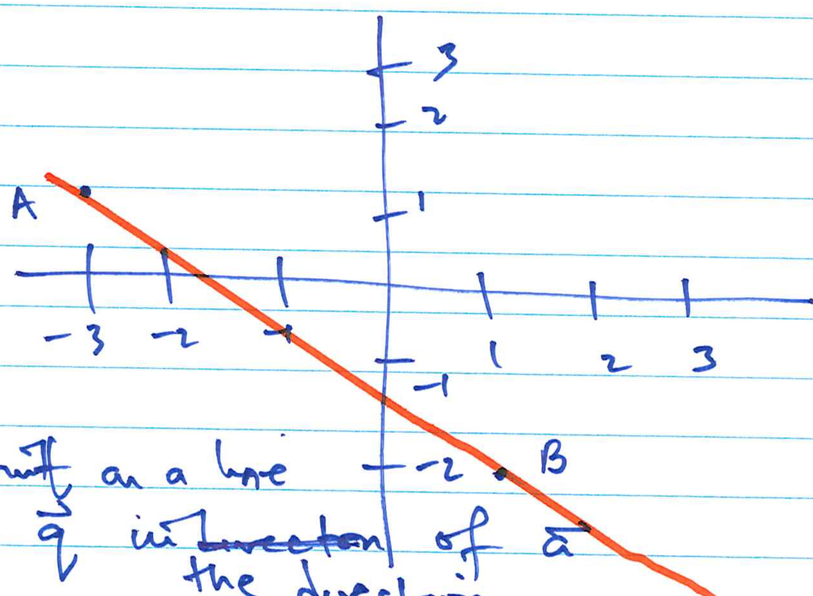
* the parametric form is not unique
* each point on the line corresponds to a unique value of t.

Example: Find ~~the~~ a ~~parametric~~ parametric form of the line that passes through the points $(-3, 1)$ and $(1, -2)$

Solution

$A = (-3, 1)$

$B = (1, -2)$



We know that each point on a line passing through point $\vec{q}$ in ~~direction~~ the direction of $\vec{a}$ is given by

$$\vec{x} = \vec{q} + t\vec{a}, \quad t \in \mathbb{R}.$$

for some value of t.

Let $\vec{q} = A$.

Let us find $\vec{a}$

$$\vec{a} = B - A = (1, -2) - (-3, 1) = (4, -3)$$

$\therefore$ Each point on the line can be written as

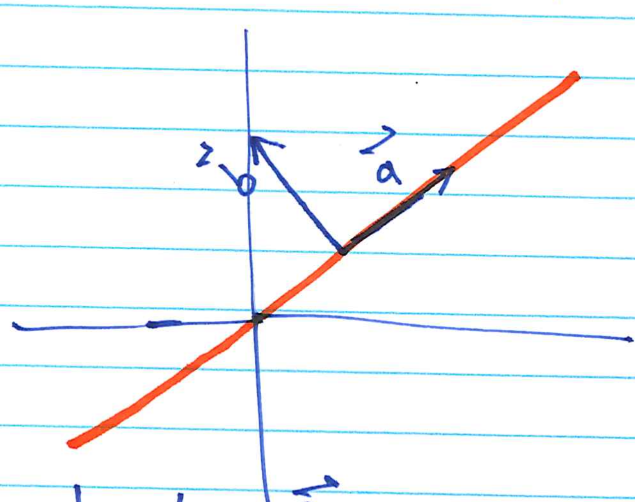
$$\vec{x} = (-3, 1) + t(4, -3), \quad \text{for some } t.$$

If $\vec{q} = (1, -2)$, then

$$\vec{x} = (1, -2) + t(4, -3), \quad \text{for some } t.$$

Equation form of a line

Let us consider a line passing through the origin and in the direction of vector $\vec{a}$.



$\vec{b}$ is orthogonal to $\vec{a}$.

Let $\vec{x}$ be a point on the line, then

$$\vec{x} \cdot \vec{b} = 0$$

Let $\vec{x} = (x_1, x_2)$ and $\vec{b} = (b_1, b_2)$

then $\vec{x} \cdot \vec{b} = (x_1, x_2) \cdot (b_1, b_2)$

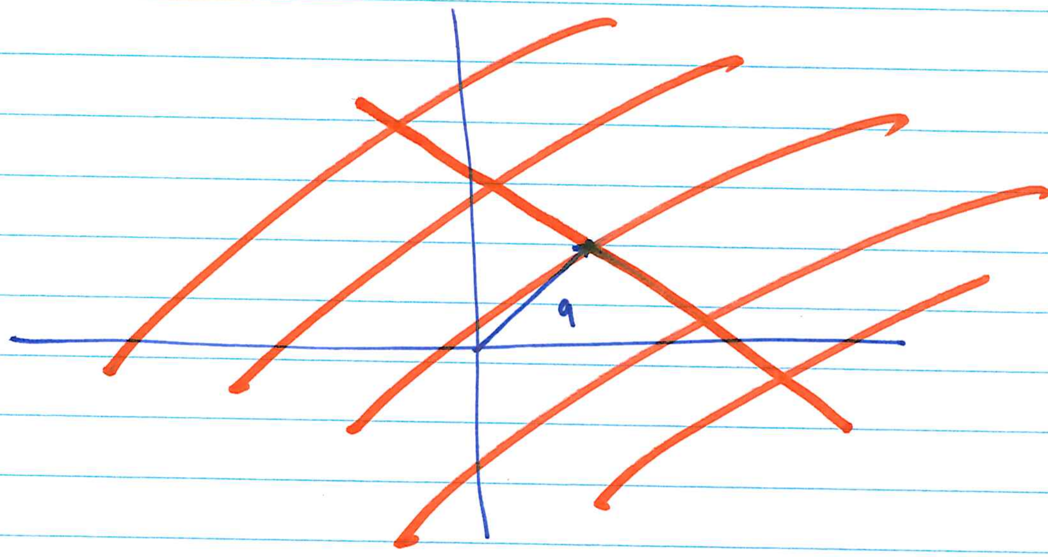
$$\vec{x} \cdot \vec{b} = x_1 b_1 + x_2 b_2$$

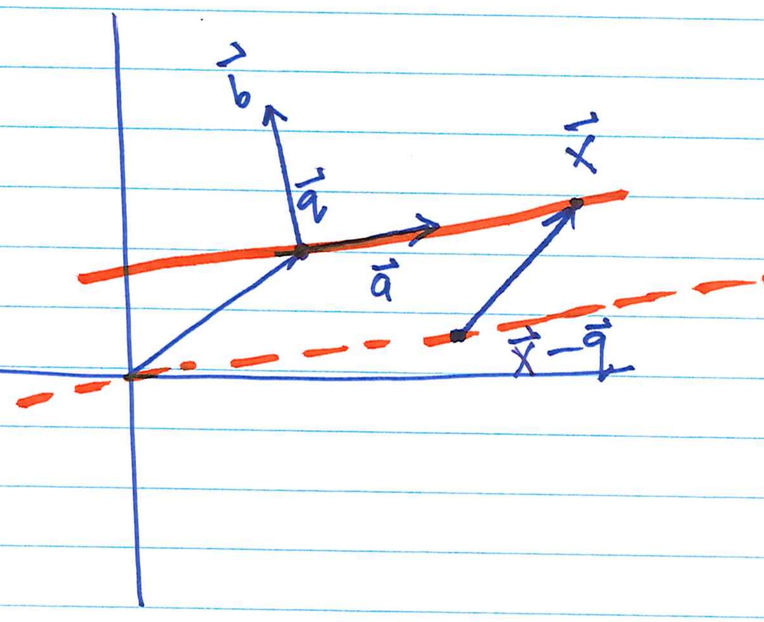
$\therefore \quad \vec{x} \cdot \vec{b} = 0$

$$\Rightarrow \quad x_1 b_1 + x_2 b_2 = 0$$

this is the equation form of a line ~~in 2D~~.
passing through the origin in 2D.

Suppose the line passes through a point $\vec{q}$





$$(\vec{x}-\vec{q})\cdot\vec{b} = 0$$

$$\vec{x}\cdot\vec{b} - \vec{q}\cdot\vec{b} = 0$$

$$\vec{x}\cdot\vec{b} = \vec{q}\cdot\vec{b}$$

If $\vec{x} = (x_1, x_2)$ and $\vec{b} = (b_1, b_2)$, then we have

$$\boxed{x_1 b_1 + x_2 b_2 = \vec{q}\cdot\vec{b}}$$

This is the equation form of a line passing through $\vec{q}$ and orthogonal to the direction of $\vec{b}$.

This is the equation form of a line passing through $\vec{q}$ and in the direction orthogonal to the direction of $\vec{b}$.

Example: Find an equation form for the line given in parametric form as

$$(1,2) + t(1,2). \longrightarrow *$$

Solution

The equation form of a line is given by

$$\vec{x} \cdot \vec{b} = \vec{q} \cdot \vec{b}$$

We know that a line in parametric form is given by $\vec{x} = \vec{q} + t\vec{a} \longrightarrow *$

$\vec{q} = (1,2) \qquad \vec{a} = (1,2)$

we need to find $\vec{b}$ and we know that $\vec{b}$ is orthogonal to $\vec{a}$

How to find a vector orthogonal to another vector !

let $\vec{a} = (a_1, a_2)$

we want find a vector $\vec{b}$ that is orthogonal to $\vec{a}$.

let $\vec{b} = (x, y)$, since $\vec{a}$ and $\vec{b}$ are orthogonal, then

$$\vec{a} \cdot \vec{b} = 0$$

$$(a_1, a_2) \cdot (x, y) = 0$$

$$a_1 x + a_2 y = 0 \quad (*)$$

$$a_1 x = -a_2 y$$

$$x = -a_2, \quad y = a_1$$

$$\vec{b} = (-a_2, a_1).$$

continuation of example!

$\vec{a} = (1, 2)$ $\vec{q} = (1, 2)$

$\vec{b} = (-2, 1)$

$\vec{X} \cdot \vec{b} = \vec{q} \cdot \vec{b}$

let $\vec{x} = (x_1, x_2)$

then $(x_1, x_2) \cdot (-2, 1) = (1, 2) \cdot (-2, 1)$

$-2x_1 + x_2 = -2 + 2 = 0$

$-2x_1 + x_2 = 0$

This ~~[illegible]~~ is an equation form of the line.

~~Example~~

Example: Find ~~the~~ an equation form for the line whose parametric form is

$$\vec{x} = (0,2) + t(2,1) \quad \text{——} \quad (1)$$

Solution

~~the eq~~ the equation is given by

$$\vec{x}\cdot\vec{b} = \vec{q}\cdot\vec{b}$$

from (1) $\vec{a} = (2,1)$, and $\vec{q} = (0,2)$.

we want to find $\vec{b}$, $\vec{b}$ is orthogonal $\vec{a}$.

$\vec{b} = (-1, 2)$

$\therefore \quad \vec{x}\cdot\vec{b} = \vec{q}\cdot\vec{b}$

$$(x_1, x_2)\cdot(-1,2) = (0,2)\cdot(-1,2)$$

$$-x_1 + 2x_2 = 4$$

Example! Find a parameteric form for the line whose equation form is given by

$$x_1 + 4x_2 = 1$$

Solution

$$x_1 + 4x_2 = 1 \quad \text{———} \quad (1)$$

we know that this is equivalent to

$$\vec{x} \cdot \vec{b} = \vec{q} \cdot \vec{b}$$

$$\vec{x} = (x_1, x_2) \text{ and } \vec{b} = (b_1, b_2)$$

$$\vec{x} \cdot \vec{b} = x_1 b_1 + x_2 b_2 = x_1 + 4x_2$$

$$\Rightarrow b_1 = 1, \quad b_2 = 4$$

$$\Rightarrow \vec{b} = (1, 4)$$

$\vec{q}$ is a point on the line

let $x_1 = 2$, Substitute into (1) to get x_2.

$$2 + 4x_2 = 1$$

$$4x_2 = 1 - 2 = -1$$

$$x_2 = -1/4$$

$$\vec{q} = (2, -1/4), \qquad \vec{a} = \vec{b}^{\perp} = (-4, 1)$$

$$\Rightarrow \vec{x} = \vec{q} + t\vec{a} = (2, -1/4) + t(-4, 1)$$

Determine if $\left(\frac{9}{2}, \frac{3}{2}, 4\right)$ is on the line.

solution

$$\vec{X} = (4,1,3) + t(1,-1,2)$$

$$= (4,1,3) + (t,-t,2t)$$

$$\vec{X} = (4+t,\ 1-t,\ 3+2t)$$

$$\left(\frac{9}{2}, \frac{3}{2}, 4\right) = \left(4+t,\ 1-t,\ 3+2t\right)$$

$$\frac{9}{2} = 4+t, \qquad \frac{3}{2} = 1-t, \qquad 3+2t = 4$$

$$t = \frac{9}{2} - 4 \qquad t = 1 - \frac{3}{2} \qquad 2t = 4-3$$

$$= \frac{1}{2} \qquad t = -\frac{1}{2} \qquad t = \frac{1}{2}$$

Since the value of the ~~one co~~ is not unique, the point is not on the line.

line in 3D

* **parametric form**

Let us consider a line that passes through a point $\vec{q}$ and in the direction of vector $\vec{a}$. Then each point on the line is given by

$$\vec{x} = \vec{q} + t\vec{a}, \quad \text{for some } t.$$

Example: Find the parametric form of line that passes through $\underbrace{(3,2,1)}_{A}$ and $\underbrace{(4,1,3)}_{B}$.

Solution:

we need $\vec{q}$ and $\vec{a}$

$$\vec{a} = B - A = (4,1,3) - (3,2,1) = (1,-1,2)$$

let $\vec{q} = (3,2,1)$, then a ~~the~~ parametric form of the line is

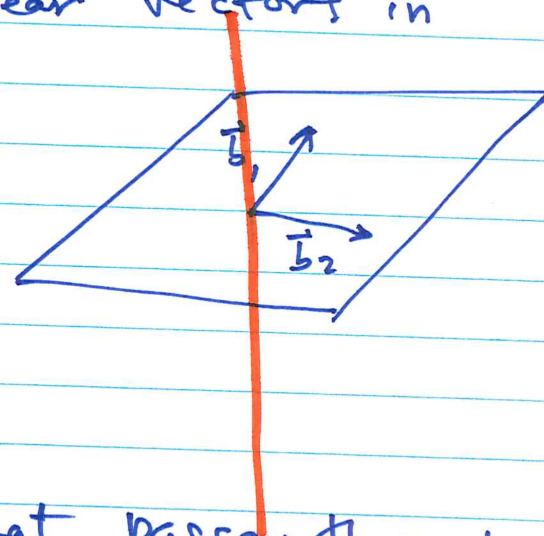
$$\vec{x} = (3,2,1) + t(1,-1,2)$$

another parametric form is

$$\vec{x} = (4,1,3) + t(1,-1,2)$$

Equation form of a line in 3D

Let $\vec{b}_1$ and $\vec{b}_2$ be non-collinear vectors in a plane



let $\vec{x}$ be a point on the line

then $\vec{x} \cdot \vec{b}_1 = 0$

and $\vec{x} \cdot \vec{b}_2 = 0$

$\therefore$ the equation form of a line ~~pa~~ that passes through the origin is given by

$$\vec{x} \cdot \vec{b}_1 = 0$$
$$\vec{x} \cdot \vec{b}_2 = 0$$

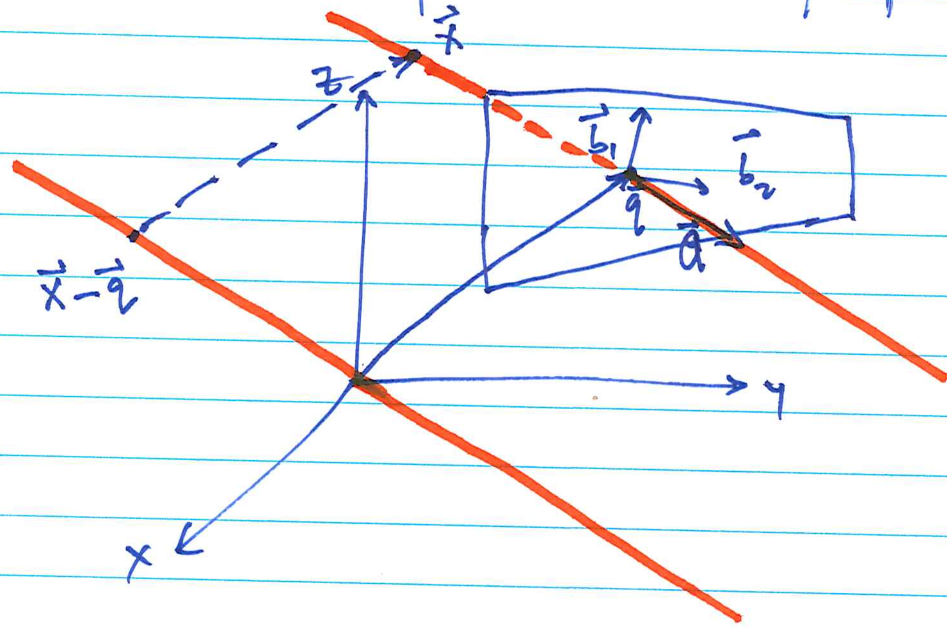
if $\vec{x} = (x_1, x_2, x_3)$, $\vec{b}_1 = (b_{11}, b_{12}, b_{13})$

$$\vec{b}_2 = (b_{21}, b_{22}, b_{23})$$

$\therefore$ $\vec{x} \cdot \vec{b}_1 = 0 \Rightarrow x_1 b_{11} + x_2 b_{12} + x_3 b_{13} = 0$

$\vec{x} \cdot \vec{b}_2 = 0 \Rightarrow x_1 b_{21} + x_2 b_{22} + x_3 b_{23} = 0$

* Suppose the line passes through point $\vec{q}$.



$$\therefore \quad (\vec{x}-\vec{q}) \cdot \vec{b}_1 = 0$$
$$(\vec{x}-\vec{q}) \cdot \vec{b}_2 = 0$$