Last day

* Eigenvalues and eigenvectors.

If we ~~an~~ have an $n \times n$ matrix A. λ is an eigenvalue of A with ~~with~~ eigenvector $\vec{x} \neq 0$ if

$$A\vec{x} = \lambda \vec{x}.$$

* The eigenvector is not uniquely determined. That is, if $\vec{x}$ is an eigenvector corresponding to eigenvalue λ, then any vector in the direction of $\vec{x}$ is also an eigenvector of λ.

* To compute the eigenvalue, we use

$$\det(A - \lambda I) = 0 \qquad \text{(characteristic equation)}$$

* to get the eigenvector, we use

$$(A - \lambda I)\vec{x} = \vec{0}$$

* We looked at an example of a matrix

$$A = \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}$$

and we ~~have~~ got $\lambda_1 = 1$ and $\lambda_2 = 2$ (repeated roots) to be the eigenvalues.

The eigenvectors are

For $\lambda_1 = 1$, $\vec{x} = \alpha \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, $\alpha \neq 0$ is a constant.

Example! take $\alpha = 1$

$$\vec{x} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$,

$$\vec{X} = \alpha_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

To get two linearly independent eigenvector ~~corresp~~ corresponding to λ_2,

* take $\alpha_1 = 0$, $\alpha_2 = 1$

$$\vec{X} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

* take $\alpha_1 = 1$, and $\alpha_2 = 0$

$$\vec{X} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, the linear independent eigenvectors of A are

$$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Example! (Repeated eigenvalue with "missing" eigenvectors).

Find the eigenvalue an eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Let λ be an eigenvalue of A

then $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) = 0$$

$\lambda = 2$ (twice).

Let $\vec{x}$ be the eigenvector of $\lambda = 2$.

then $(A - \lambda I)\vec{x} = \vec{0}$

$$\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We observe that the repeated eigenvalue gave us only one eigenvector.
We can use the generalized eigenvector technique to find the remaining linearly independent eigenvector. This method will not be covered in this course.

Complex eigenvalues and eigenvectors

Example:

Consider the matrix of rotation by $\pi/2$ in clockwise direction. Find the eigenvalues and eigenvectors of the matrix

Solution

The rotation matrix is given by

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\theta = -\pi/2$$

$$A = \begin{pmatrix} \cos(-\pi/2) & -\sin(-\pi/2) \\ \sin(-\pi/2) & \cos(-\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If λ is an eigenvalue of A, then

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\lambda = \pm\sqrt{-1} = \pm i$$

$$\lambda_1 = +i \quad , \quad \lambda_2 = -i$$

For $\lambda_1 = i$,

$$(A - \lambda_1 I)\vec{x} = \vec{0}$$

$$\Rightarrow \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 = R_2 + iR_1$$

$$-1 + i(-i) = -1 + 1 = 0$$

$$-i + i = 0$$

$$-ix_1 + x_2 = 0 \qquad\qquad \frac{x_2 \times i}{i \times i} =$$

$$-ix_1 = -x_2$$

$$x_1 = \frac{x_2}{i} = -ix_2$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}$$, $x_2 \neq 0$ if free.

take $x_2 = 1$, $\vec{X} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

For $\lambda_2 = -i$, $(A - \lambda_2 I)\vec{X} = \vec{0}$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \frac{-1 \times i}{i \times i} \quad \frac{-i}{i}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 = R_2 - iR_1$$

$$ix_1 + x_2 = 0$$

$$ix_1 = -x_2$$

$$x_1 = -\frac{x_2}{i} = +ix_2$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$$, $x_2 \neq 0$ is free.

Remarks (some properties of complex eigenvalues and eigenvectors)

* If $\lambda \in \mathbb{C}$ is an eigenvalue of an $n \times n$ matrix, then the complex conjugate of λ, $\bar{\lambda}$ is also an eigenvalue of the matrix

* Let A be an $n \times n$ real matrix (a matrix with real entries)
If $\lambda \in \mathbb{C}$ is an eigenvalue of A, with eigenvector $\vec{x} \neq 0$. Then

$$A\vec{x} = \lambda \vec{x} \quad \text{---} \quad (1)$$

Take the complex conjugate of (1).

$$\overline{A\vec{x}} = \overline{\lambda \vec{x}}$$

$$\bar{A}\,\bar{\vec{x}} = \bar{\lambda}\,\bar{\vec{x}} \quad \text{---} \quad (2)$$

But A is a real matrix, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

$$\Rightarrow \quad \bar{A} = A$$

$\Rightarrow$ ② becomes

$$A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$$

let $\overline{\vec{x}} = \vec{y}$

then $A\vec{y} = \overline{\lambda}\vec{y}$

$\Rightarrow$ $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\vec{y}$.

Illustration

Consider the previous example:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A \text{ is a real matrix.}$$

with ~~eigenvect~~ eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$

$\Rightarrow$ $\lambda_2 = \overline{\lambda_1}$ ✓.

The eigenvector of $\lambda_1 = i$ is $\vec{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$\overline{\vec{x}} = \overline{\begin{pmatrix} -i \\ 1 \end{pmatrix}} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \vec{y} \quad ✓$$

which is the ~~complex con~~ eigenvector of $\lambda_2 = -i$.

power of a matrix

~~Reca~~ Recall, that ~~we~~ when we studied random walk, we needed to take the n^{th} power of ~~a matrix~~ the transition matrix, P to get the state of the system after n time steps.
that is,
If $\vec{X}_0$ is the initial state of the system, then $\vec{X}_n = P^n \vec{X}_0$.

We shall use eigen analysis is to compute the power of the matrix P.

<u>Example:</u> Consider a network with the transition matrix given by

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix}$$

~~with~~ with initial condition $\vec{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

We computed the eigenvalues and eigenvectors to be

$$\lambda_1 = 1, \quad \vec{Y}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 1, \quad \vec{Y}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{\sqrt{2}}, \quad \vec{Y}_3 = \begin{pmatrix} 1-\sqrt{3} \\ \frac{2}{3}(1-\sqrt{3}) \\ \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}$$

$$\lambda_4 = -\frac{1}{\sqrt{3}}, \quad \vec{Y}_4 = \begin{pmatrix} 1+\sqrt{3} \\ -\frac{2}{\sqrt{3}}(1+\sqrt{3}) \\ -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}$$

The eigenvectors $\vec{Y}_1$, $\vec{Y}_2$, $\vec{Y}_3$, and $\vec{Y}_4$ are linearly independent, therefore, they form a basis for $\mathbb{R}^4$.

Let us write $\vec{X}_0$ as a linear combination of these vectors.

$$\vec{X}_0 = \alpha_1 \vec{Y}_1 + \alpha_2 \vec{Y}_2 + \alpha_3 \vec{Y}_3 + \alpha_4 \vec{Y}_4$$

this gives a system of equation which can be written in matrix form as

$$T \vec{\alpha} = \vec{X}_0$$

where
$$T = \begin{pmatrix} | & | & | & | \\ \vec{Y}_1 & \vec{Y}_2 & \vec{Y}_3 & \vec{Y}_4 \\ | & | & | & | \end{pmatrix}$$

and
$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

We can ~~use~~ solve this system (say, with MATLAB) to get

$$\vec{\alpha} = \begin{pmatrix} 1/2 \\ 1/2 \\ -0.6830 \\ 0.1830 \end{pmatrix}$$

To get the state of the system after the first step,

$$\vec{X}_1 = P\vec{X}_0$$

$$= P\left(\alpha_1 \vec{Y}_1 + \alpha_2 \vec{Y}_2 + \alpha_3 \vec{Y}_3 + \alpha_4 \vec{Y}_4\right)$$

$$= \alpha_1 (P\vec{Y}_1) + \alpha_2 (P\vec{Y}_2) + \alpha_3 (P\vec{Y}_3) + \alpha_4 (P\vec{Y}_4)$$

but the vectors $\vec{Y}_1, \ldots, \vec{Y}_4$ are eigenvectors of P.

$$\Rightarrow \quad P\vec{Y}_i = \lambda_i \vec{Y}_i \quad \text{for} \quad i = 1, 2, 3, 4.$$

$$\therefore \vec{X}_1 = \alpha_1 (\lambda_1 \vec{Y}_1) + \alpha_2 (\lambda_2 \vec{Y}_2) + \alpha_3 (\lambda_3 \vec{Y}_3) + \alpha_4 (\lambda_4 \vec{Y}_4)$$

Substituting the eigenvalues,

$$\vec{X}_1 = 1 \cdot \alpha_1 \vec{Y}_1 + 1 \cdot \alpha_2 \vec{Y}_2 + \frac{1}{\sqrt{3}} \alpha_3 \vec{Y}_3 - \frac{1}{\sqrt{3}} \alpha_4 \vec{Y}_4.$$

For $\vec{X}_2$,

$$\vec{X}_2 = P\vec{X}_1$$

$$= P\left(\alpha_1\vec{Y}_1 + \alpha_2\vec{Y}_2 + \frac{1}{\sqrt{3}}\alpha_3\vec{Y}_3 - \frac{1}{\sqrt{3}}\alpha_4\vec{Y}_4\right)$$

$$= \alpha_1(P\vec{Y}_1) + \alpha_2(P\vec{Y}_2) + \frac{1}{\sqrt{3}}\alpha_3(P\vec{Y}_3) - \frac{1}{\sqrt{3}}\alpha_4(P\vec{Y}_4)$$

$$\vdots$$

$$\vec{X}_2 = \alpha_1\vec{Y}_1 + \alpha_2\vec{Y}_2 + \left(\frac{1}{\sqrt{3}}\right)^2\alpha_3\vec{Y}_3 + \left(-\frac{1}{\sqrt{3}}\right)^2\alpha_4\vec{Y}_4$$

Continuing this way,

$$\vec{X}_n = \alpha_1\vec{Y}_1 + \alpha_2\vec{Y}_2 + \left(\frac{1}{\sqrt{3}}\right)^n\alpha_3\vec{Y}_3 + \left(-\frac{1}{\sqrt{3}}\right)^n\alpha_4\vec{Y}_4$$

Example: consider a network with transition matrix

$$P = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{pmatrix}$$

and $\vec{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We have the eigen analysis;

$$\lambda_1 = 1 \quad , \quad \vec{Y}_1 = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}$$

$$\lambda_2 = 1/5 \quad , \quad \vec{Y}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1/5 \quad , \quad \vec{Y}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

we can write

$$\vec{X}_0 = \alpha_1 \vec{Y}_1 + \alpha_2 \vec{Y}_2 + \alpha_3 \vec{Y}_3$$

and in matrix form,

$$T \vec{\alpha} = \vec{X}_0$$

We solve the system ~~for~~ ~~two~~ and got

$$\vec{\alpha} = \begin{pmatrix} 1 \\ -1/4 \\ 1/12 \end{pmatrix}$$

We have

$$\vec{X}_0 = \alpha_1 \vec{Y}_1 + \alpha_2 \vec{Y}_2 + \alpha_3 \vec{Y}_3$$

$$\vec{X}_0 = 1 \cdot \vec{Y}_1 + (-1/4) \vec{Y}_2 + \frac{1}{12} \vec{Y}_3$$

To get $\vec{X}_1$,

$$\begin{aligned} \vec{X}_1 &= P \vec{X}_0 \\ &= P \left(\vec{Y}_1 - 1/4 \vec{Y}_2 + 1/12 \vec{Y}_3 \right) \\ &= P\vec{Y}_1 - 1/4 (P\vec{Y}_2) + 1/12 (P\vec{Y}_3) \\ &= \lambda_1 \vec{Y}_1 - 1/4 (\lambda_2 \vec{Y}_2) + \frac{1}{12} (\lambda_3 \vec{Y}_3) \end{aligned}$$

we know that $\lambda_1 = 1$, $\lambda_2 = 1/5$, $\lambda_3 = -1/5$

$$\vec{X}_1 = 1 \cdot \vec{Y}_1 - \frac{1}{4} (1/5) \vec{Y}_2 + \frac{1}{12} (-1/5) \vec{Y}_3$$

$\vdots$

$$\vec{X}_n = (1)^n \vec{Y}_1 - \frac{1}{4} (1/5)^n \vec{Y}_2 + (1/12)(-1/5)^n \vec{Y}_3$$

what happens to the system as $n \to \infty$?

$$\lim_{n\to\infty} \vec{X}_n = \lim_{n\to\infty} \left(\vec{Y}_1 + \tfrac{1}{4}\left(\tfrac{1}{5}\right)^n \vec{Y}_2 + \left(\tfrac{1}{12}\right)\left(-\tfrac{1}{5}\right)^n \vec{Y}_3 \right)$$

But $\left(\tfrac{1}{5}\right)^n \to 0$ and $\left(-\tfrac{1}{5}\right)^n \to 0$

as $n \to \infty$.

$$\therefore \lim_{n\to\infty} \vec{X}_n = \vec{Y}_1$$

The system ~~can~~ will converge to $\vec{Y}_1$ irrespective of the initial condition.

If this happens, $\vec{Y}_1$ is called the equilibrium probability of the system.