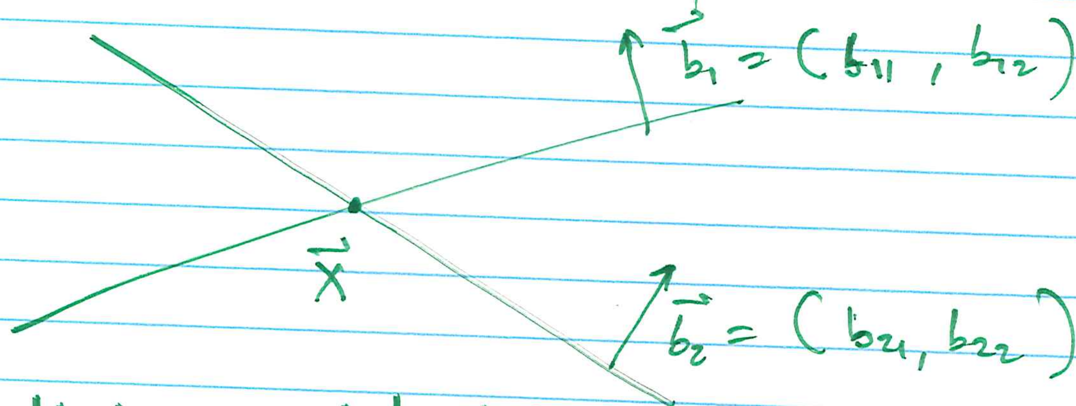
Last day

Consider the 2D system

$$b_{11} x_1 + b_{12} x_2 = c_1$$
$$b_{21} x_1 + b_{22} x_2 = c_2$$

then the solution of this can be seen has the point of intersection of 2 lines.

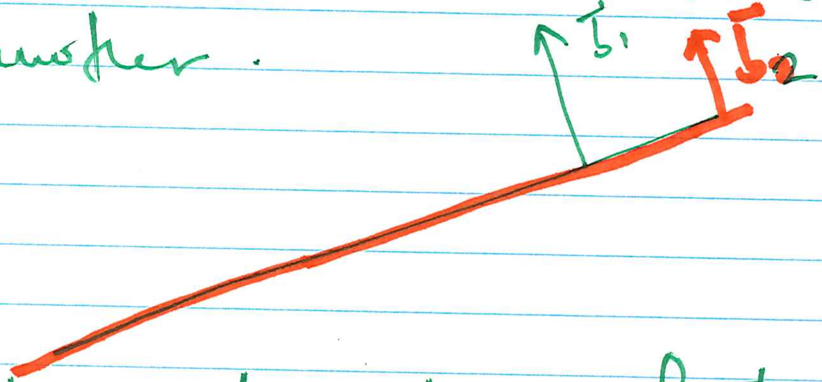


Case I : Unique solution

If the vectors $\vec{b_1}$ and $\vec{b_2}$ are non-collinear, then the two lines are not parallel and so they intersect at only one point which is the solution of the system of equations.
$\therefore$ the system as a unique solution.

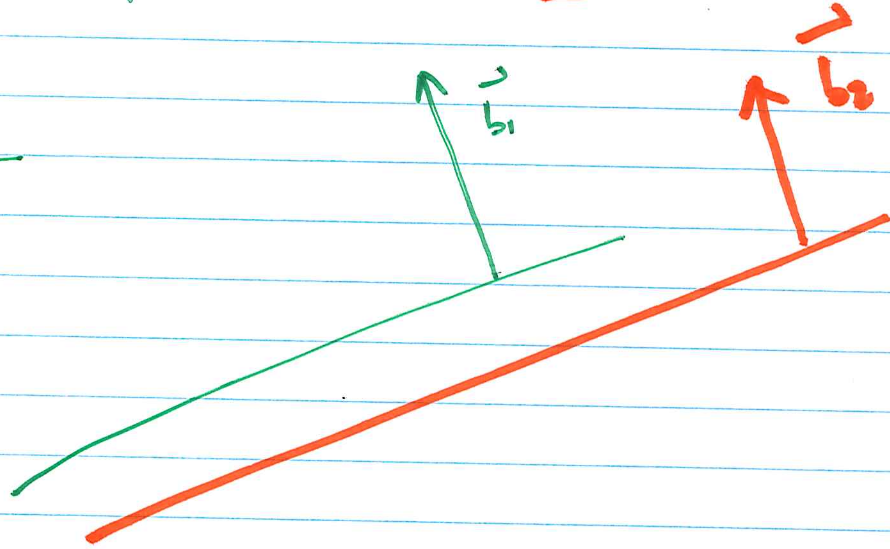
Case II :

If the two lines are scalar multiple of one another.



Then the system has infinite number of solutions

Case III :



the system has no solution.

Consider the system

$$b_{11}x_1 + b_{12}x_2 = c_1$$
$$b_{21}x_1 + b_{22}x_2 = c_2.$$

Let $A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be the coefficient matrix of the system

Let $\vec{b}_1 = (b_{11}, b_{12})$, $\vec{b}_2 = (b_{21}, b_{22})$

then if $\vec{b}_1$ and $\vec{b}_2$ are scalar multiple of one another, then

$$\det(A) = 0$$

then the system has no unique solution.

$\Rightarrow$ If $\det(A) \neq 0$, then the system has a unique solution.

parallel lines

Example: Determine whether the following system has no ~~soln~~ solution, unique solution, or infinitely many solutions.

(a) $3x + 4y = 6$

$x + y = 2$

$A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$, $\det(A) = 3 - 4 = -1 \neq 0$

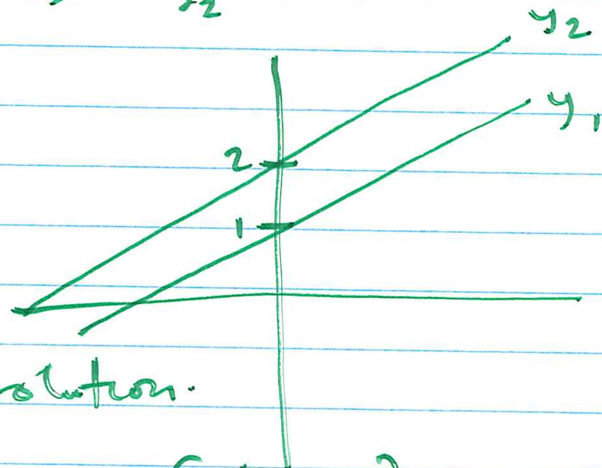
$\Rightarrow$ the system has a unique solution.

(b) $-2x + y = 1 \Rightarrow y_1 = 1 + 2x$

$-2x + y = 2 \Rightarrow y_2 = 2 + 2x$

$A = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}$

$\det(A) = -2 + 2 = 0$



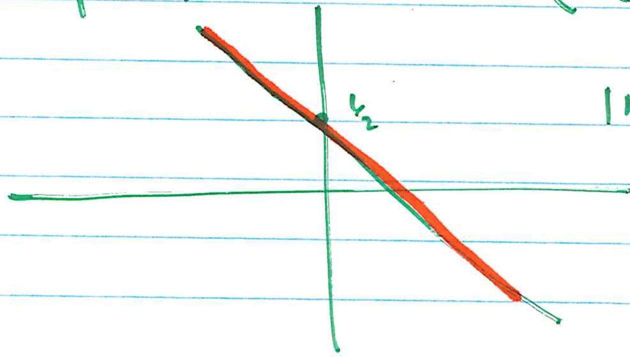
$\therefore$ the system has no solution.

(c) ~~$2x$~~ $x + 2y = 1$

$x + 2y = 1$

$\Rightarrow A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $\det(A) = 0$

$y = \frac{1}{2} - \frac{x}{2}$



Infinite many solution

System in 3D

$$b_{11} x_1 + b_{12} x_2 + b_{13} x_3 = c_1$$

$$b_{21} x_1 + b_{22} x_2 + b_{23} x_3 = c_2$$

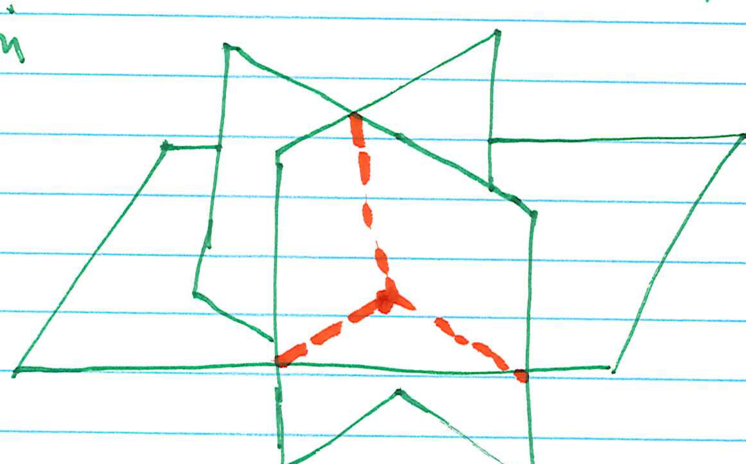
$$b_{31} x_1 + b_{32} x_2 + b_{33} x_3 = c_3$$

$\vec{b_1}$, $\vec{b_2}$ and $\vec{b_3}$ are vectors that don't lie on the ~~system~~ plane the same plane.

If $\vec{x}$ is a point of intersection of the 3 planes, then $\vec{x}$ is a solution of the system.

Unique solution

If the vectors $\vec{b_1} = (b_{11}, b_{12}, b_{13})$, $\vec{b_2} = (b_{21}, b_{22}, b_{23})$ and $\vec{b_3} = (b_{31}, b_{32}, b_{33})$ are not lying on the same plane, then the system has a unique solution

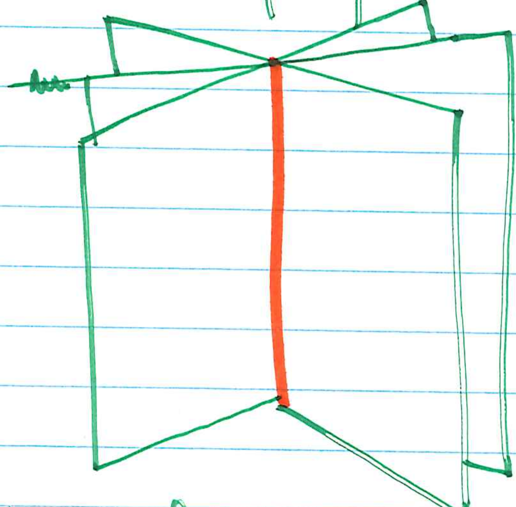


Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

If $\det(B) \neq 0$, then the system has a unique solution.

Other cases

① If $\det(B) = 0$, and the vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$ are not lying on the same plane.



* No unique solution

The solution of the system lies on the ~~common~~ line of intersection of the 3 planes.

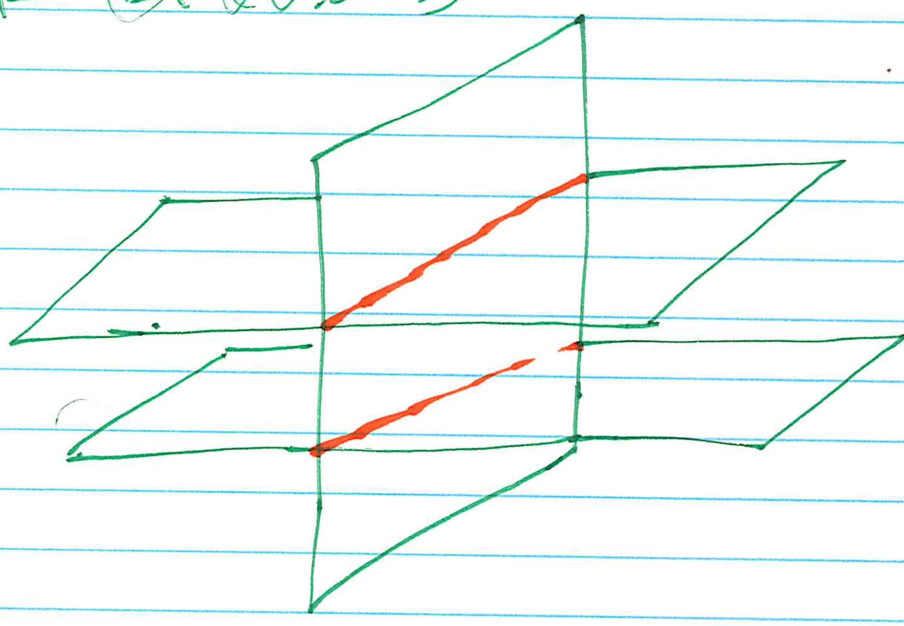
$$\vec{x} = \vec{q} + t\vec{a}.$$

(ii) If $\det(B) = 0$, it could be that the planes are the same, and so the set of solution of the system is the plane. This plane can be written in parametric form as

$$\vec{x} = \vec{q} + t_1 \vec{a_1} + t_2 \vec{a_2}$$

the system has infinitely many solutions.

(iii) ~~If det(B)=0~~



the ~~sys~~ system has no solution

Example:

$$2x_1 + x_2 + x_3 = 6$$
$$x_1 + \quad + 2x_3 = 3$$
$$x_1 + 2x_2 + 3x_3 = 4$$

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\det(B) = 2(-4) - 1(3-2) + 1(2-0)$$
$$= -8 - 1 + 2 = -7 \neq 0$$

Since $\det(B) \neq 0$, the system has a unique solution.

Example:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + \quad + x_3 = 2$$
$$x_1 + 2x_2 + x_3 = 1$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \det(B) = -2 + 0 + 2 = 0.$$

The solution of the system is not unique.

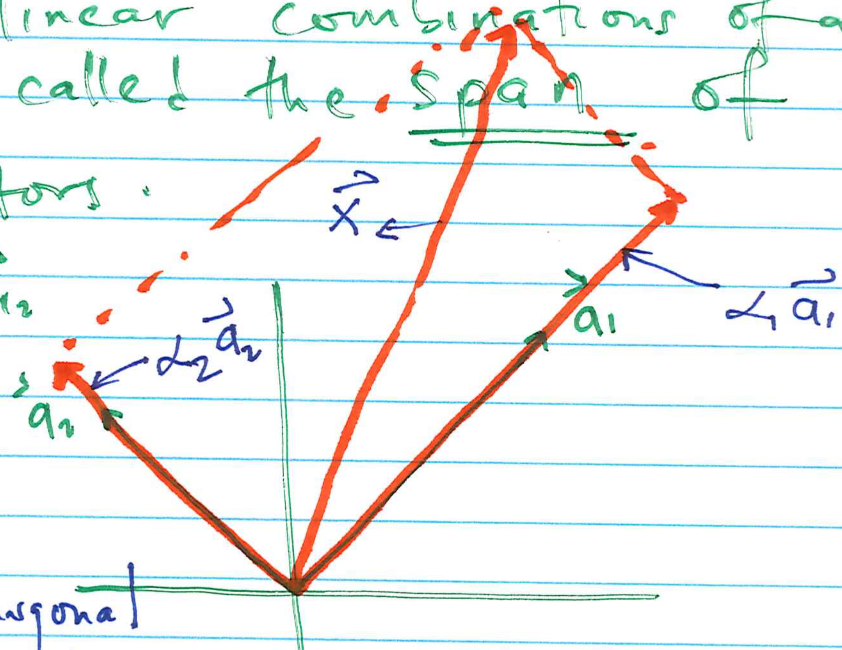
linear combination of vectors

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be vectors of the same dimension, and not lying on the same plane, a linear combination of $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is $\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \ldots + \alpha_n \vec{a}_n$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

The set of all linear combinations of a set of vectors is called the <u>span</u> of the set of vectors.

In 2D $\vec{a}_1$ and $\vec{a}_2$



$\vec{x} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2$

If $\vec{a}_1$ and $\vec{a}_2$ are orthogonal and with unit length, then we can write $\vec{x} = (\alpha_1, \alpha_2)$

$\exists$ — there exist

linear dependence and independence.

A collection of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is called linearly dependent if $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_n \vec{a}_n = \vec{0}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zeros.

If a collection of vectors in not linearly dependent, then it is linearly independent.

A collection of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is linearly independent if the only way we can have

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_n \vec{a}_n = 0$$

is that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$$

BASIS

A collection of n linearly independent vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ in $\mathbb{R}^n$ dimensional space is called a basis. ~~of $\mathbb{R}^n$~~

Suppose $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ is a basis for $\mathbb{R}^n$, then $\vec{X} \in \mathbb{R}^n$ can be written uniquely as a linear combination of the basis vectors.

i.e $\vec{X} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_n \vec{a}_n$

for some $\alpha_1, \alpha_2, \dots, \alpha_n$ scalars.

dimension of a span

The dimension of a span of a set Ω of linearly independent vectors is equal to the number of vectors in Ω (the set of vectors).

Example $\Omega = \{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ vectors are linearly independent

dimension of span of $\Omega = 4$