Last day

* linear transformations

* Matrix representation of transformations.

If $T: \mathbb{R}^2 \to \mathbb{R}^2$, then the matrix of the transformation is

$$T = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are the standard bases for $\mathbb{R}^2$.

* Composition of linear transformations.

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $S: \mathbb{R}^p \longrightarrow \mathbb{R}^m$

Then $S(T(\vec{x}))$ is a composition of the two transformations. And this ~~the~~ transformation is linear.

* $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

In general,
$$S(T(\vec{x})) \neq T(S(\vec{x}))$$

we can easily see this using matrix ~~ma~~ ~~multi~~ multiplication.

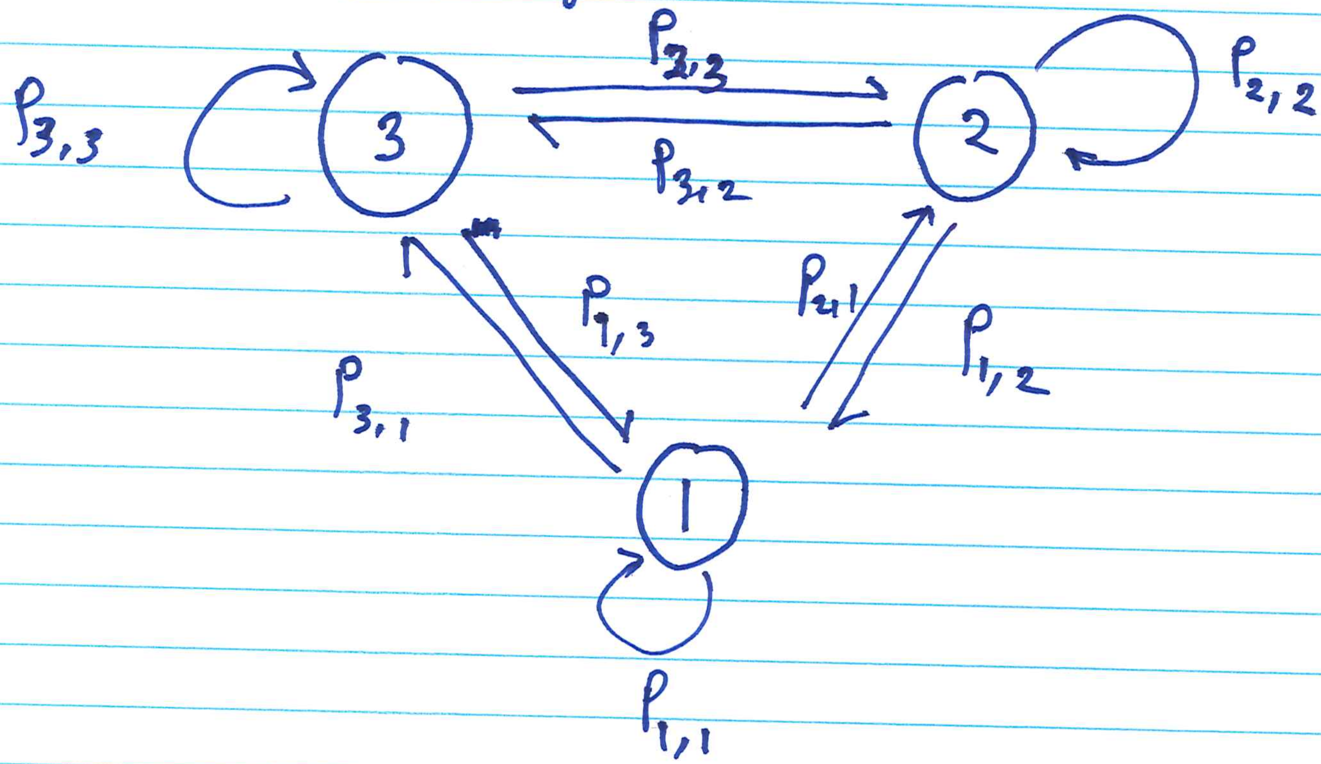
$$\hat{S}\hat{T} \neq \hat{T}\hat{S}$$

where $\hat{S}$ is the matrix for the transformation S

and $\hat{T}$ is the matrix for the transformation T.

RANDOM WALKS

Consider a system with 3 states, suppose a random walker can move from one location to another with a certain probability.
Let $P_{i,j}$ be the probability that a random walker at location j will go to location i.



Consider location 2: suppose the random walker is at this location and he takes a step, there are 3 possibilities.

- he remains at location 2 with probability $P_{2,2}$

- he moves to 3 with probability $P_{3,2}$
- moves to 1 with probability $P_{1,2}$

and

$$P_{1,2} + P_{2,2} + P_{3,2} = 1$$

In general,

$$P_{1,j} + P_{2,j} + P_{3,j} = \sum_{i=1}^{3} P_{i,j} = 1.$$

for $j = 1, 2, 3$.

Let $x_{n,j}$ be the probability that the random walker is at location j at <u>time n</u>,

$$\vec{X}_n = \begin{pmatrix} x_{n,1} \\ x_{n,2} \\ x_{n,3} \end{pmatrix}$$

and if we have $\vec{X}_n$, then we can construct $\vec{X}_{n+1} = \begin{pmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{pmatrix}$

$$\cancel{x_{n+1,1} = x_{n,1} P_{1,1} + x_{n,2} P_{2,1} + x_{n,3} P_{3,1}}$$

$$x_{n+1,1} = x_{n,1} P_{1,1} + x_{n,2} P_{1,2} + x_{n,3} P_{1,3} \quad \text{— (1)}$$

~~An~~ similarly,

$$x_{n+1,2} = x_{n,1} P_{2,1} + x_{n,2} P_{2,2} + x_{n,3} P_{2,3} \quad \text{— (2)}$$

$$x_{n+1,3} = x_{n,1} P_{3,1} + x_{n,2} P_{3,2} + x_{n,3} P_{3,3} \quad \text{— (3)}$$

We have a system of equations which can be written in matrix form as

$$\begin{pmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{pmatrix} = \underbrace{\begin{pmatrix} P_{1,1} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}}_{P} \begin{pmatrix} x_{n,1} \\ x_{n,2} \\ x_{n,3} \end{pmatrix}$$

$$\vec{X}_{n+1} = P \vec{X}_n .$$

Remark

- ~~The~~ The entries of each column of matrix P must sum to 1.

Suppose $\vec{X}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ It may be e.g. $\begin{pmatrix} 0.2 \\ 0.1 \\ 0.7 \end{pmatrix}$

$$\vec{X}_1 = P\vec{X}_0$$

$$\vec{X}_2 = P\vec{X}_1 = P\left(P\vec{X}_0\right) = P^2\vec{X}_0$$

$$\vdots$$

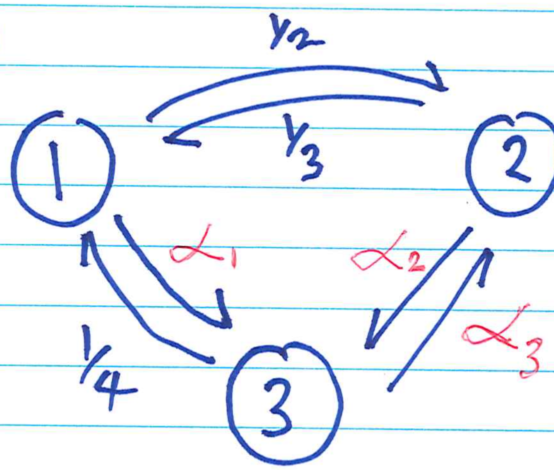
$$\vec{X}_K = P^K \vec{X}_0$$

Remarks

1. The sum of the entries of $\vec{X}_K$ must equal 1.

2. This idea can be generalized to a system with arbitrary number of locations.

Example: Consider a random walk with 3 states, as given below



Write down the matrix P of the system and find the probability that the random walker is at location 1 after 5 steps.

$$\vec{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$P_{i,j}$$

<u>Solution</u>

$$
P = \begin{array}{c} \\ \\ \\ \\ \end{array}
\begin{array}{ccc|c}
1 & 2 & 3 & \\
\end{array}
$$

$$
P = \begin{array}{cc}
\begin{array}{ccc} 1 & 2 & 3 \end{array} & \\
\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{4} \\ 0.5 & 0 & \frac{3}{4} \\ 0.5 & \frac{2}{3} & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

Let us find $\vec{X}_5$

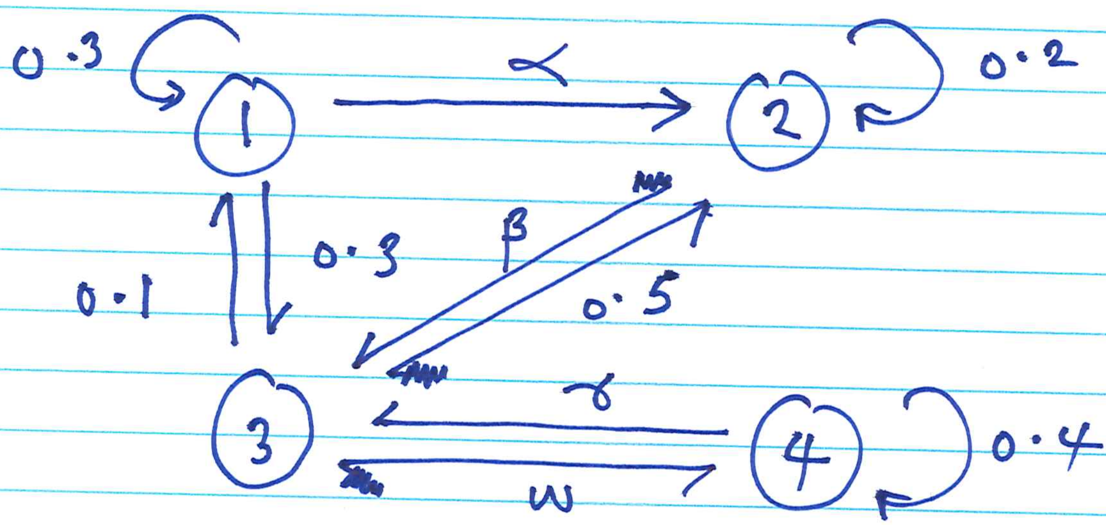
$$\vec{X}_5 = P^5 \vec{X}_0$$

using MATLAB,

$$\vec{X}_5 = \begin{pmatrix} 0.2257 \\ 0.3915 \\ 0.3828 \end{pmatrix}$$

The probability that the random walker will be at location 1 after 5 steps is <u>0.2257</u>.

Example: Consider the system given below



Construct the matrix of the network and find the state of the network after 10 steps with

$$\vec{X}_0 = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

Solution

$$P = \begin{array}{cc} \begin{pmatrix} 0.3 & 0 & 0.1 & 0 \\ 0.4 & 0.2 & 0.5 & 0 \\ 0.3 & 0.8 & 0 & 0.6 \\ 0 & 0 & 0.4 & 0.4 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \end{array}$$

$$\vec{X}_{10} = P^{10}\,\vec{X}_0$$

and this gives

$$\vec{X}_{10} = \begin{pmatrix} 0.0571 \\ 0.2785 \\ 0.3979 \\ 0.2665 \end{pmatrix}$$

The Transpose

Let A be an $m \times n$ matrix, the transpose of A is the matrix whose rows are the columns of A (in the same order). It is denoted by A^T.

Example:

Let $A = \begin{pmatrix} 1 & 2 & 4 & 7 & 5 \\ 2 & 3 & 4 & 1 & 0 \\ 0 & 1 & 7 & 2 & 8 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 4 & 7 \\ 7 & 1 & 2 \\ 5 & 0 & 8 \end{pmatrix}$$

NB

If A is square matrix, and $A^T = A$, then A is called a symmetric matrix.

Remarks:

① Let A be an $m \times n$ matrix. Then for any vector $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$,

$$\vec{y} \cdot (A\vec{x}) = (A^T \vec{y}) \cdot \vec{x}$$

(ii) For two matrices A and B,

$$(AB)^T = B^T A^T$$

~~MATRIX~~ **MATRIX INVERSES**

Definition:

Let A be an $n \times n$ matrix. A matrix B is ~~an~~ the inverse of A if

$$BA = I$$

denoted by $B = A^{-1}$

Consider a system of equations given by

$$Ax = b$$

Let $B = A^{-1}$, and multiply the system by B from the left.

$$B(Ax) = Bb$$

$$A^{-1}(Ax) = A^{-1}b$$

$$(A^{-1}A)x = A^{-1}b \qquad \text{(we have used the associative property here)}$$

$$I x = A^{-1} b$$

$$x = A^{-1} b$$

Remark

- the inverse of a matrix is unique
- If B is the inverse of A, then

$$AB = BA = I.$$

Relating inverse of a matrix to system of equations

~~Let~~

Let A be an $n \times n$ matrix. The following statements are equivalent:

(i) A is invertible.

(ii) The system of equations $A\vec{x} = b$ always has a unique solution.

(iii) The equation $A\vec{x} = 0$ has only the trivial solution.

(iv) The rank of A is n.

(v) The row echelon form of A is of the form