Last day

* application of eigen-analysis to random walk
* Some properties of a transition matrix
  * If P is a transition matrix, then
    - All eigenvalues of P satisfy $|\lambda| \leq 1$.
    - P has an eigenvalue $\lambda = 1$. If $\exists$ only one of such eigenvalues, then the corresponding eigenvector is a scalar multiple of the equilibrium probability of the random walk.
    - the remaining eigenvalues of P satisfy $|\lambda| < 1$. The eigenvectors corresponding to these eigenvalues have entries that sum to zero.

Application of eigen-analysis to differential equations (D.E.)

Review of scalar D.E

Let us consider the simplest form of D.E.

$$\frac{dy}{dt} = \lambda y$$

If given an initial value for y, that is, if given $y(0) = y_0$. Then we have an initial value problem ~~and~~ (IVP) and the solution to this problem is unique.

Consider the IVP.

$$\frac{dy}{dt} = \lambda y(t) \quad , \quad y(0) = y_0$$

Solving the DE, we have

$$y(t) = c\, e^{\lambda t}$$

if we apply the initial condition, we get $c = y_0$

$$\Rightarrow y(t) = y_0 e^{\lambda t}$$

Let $\lambda \in \mathbb{R}$.

- if $\lambda > 0$, exponential growth.
- if $\lambda < 0$, exponential decay
- if $\lambda = 0$, the solution is a constant ($y = y_0$)

Let $\lambda \in \mathbb{C}$, say $\lambda = \alpha + i\beta$.

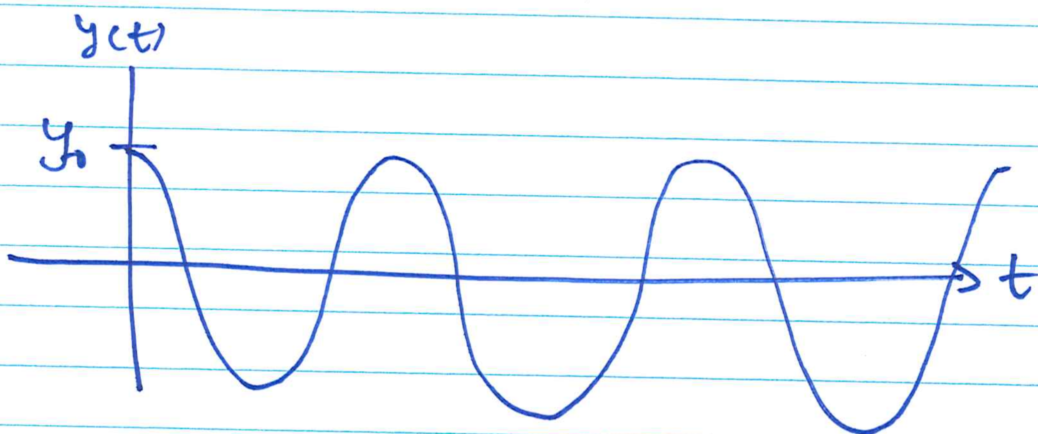
$$y(t) = y_0 e^{\lambda t} = y_0 e^{(\alpha + i\beta)t}$$

$$= y_0 e^{\alpha t}\left(\cos(\beta t) + i \sin(\beta t)\right)$$

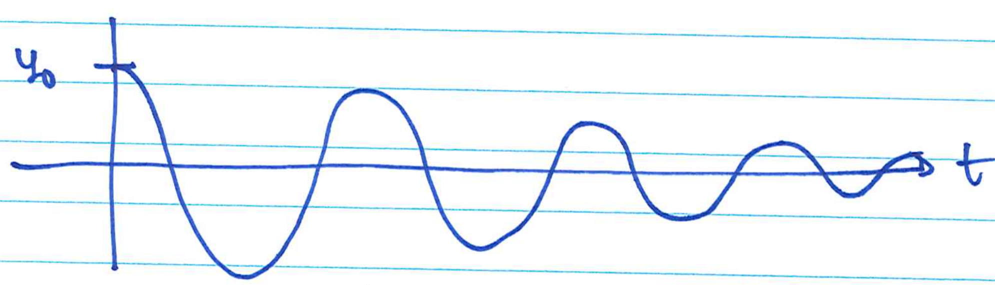
$$y(t) = y_0 e^{\alpha t} \left(\cos(\beta t) + i \sin(\beta t) \right)$$

- if $\alpha = 0$, oscillations

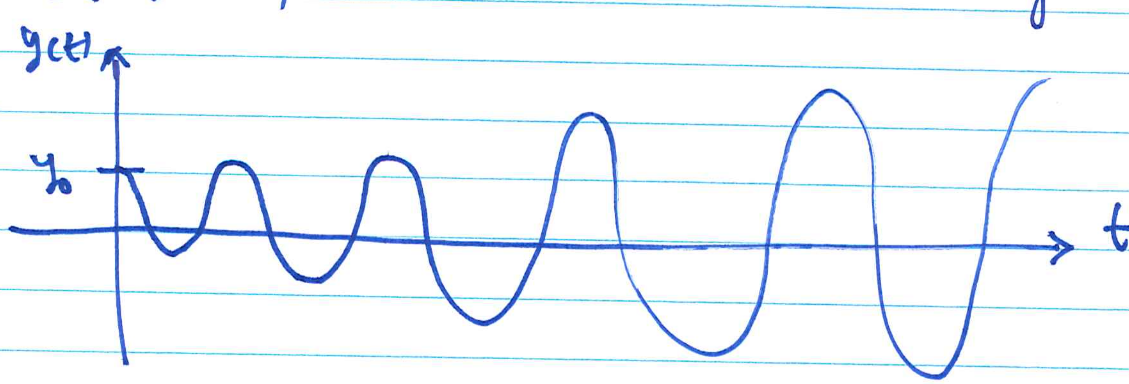
$$y(t) = y_0 \left(\cos(\beta t) + i \sin(\beta t) \right)$$



- $\alpha < 0$, damped oscillation



- $\alpha > 0$, oscillations that grows.



SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Consider the following system of linear ODE,

$$\left.\begin{aligned} y_1'(t) &= a_{11}\, y_1(t) + a_{12}\, y_2(t) \\ y_2'(t) &= a_{21}\, y_1(t) + a_{22}\, y_2(t) . \end{aligned}\right\} \quad \text{—— (1)}$$

Let $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

$$\frac{d}{dt}\left(\vec{Y}(t)\right) = \frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}$$

we write (1) in matrix form

$$\vec{Y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\Rightarrow \quad \vec{Y}'(t) = A\,\vec{Y}(t) \quad \text{—— (2)}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

~~This is the gen~~

2 ~~is~~ is the matrix form of 1.

Recall, ~~from~~ for Scalar ODE, (Ordinary Differential Equation)

$$du/dt = \alpha\, u(t)$$

the ~~gen~~ solution is $u(t) = c_1 e^{\alpha t}$.

Let us use this idea to construct a solution for the system ~~in~~ 2

Let $\vec{Y}(t) = c\, e^{\lambda t}\, \vec{x}$, $\vec{x}$ is a vector independent of t.

$$\vec{Y}'(t) = \lambda c\, e^{\lambda t} \vec{x}$$

Substitute into 2

$$\lambda \cancel{c}\, e^{\lambda t} \vec{x} = A \cancel{c}\, e^{\lambda t} \vec{x}$$

$$\cancel{e^{\lambda t}}\, \lambda \vec{x} = A\vec{x}\, \cancel{e^{\lambda t}}$$

$$\Rightarrow \quad A\vec{x} = \lambda \vec{x} \qquad \text{(eigenvalue problem)}$$

Thus,

$$\vec{y}(t) = C e^{\lambda t} \vec{x} \qquad (C\text{, constant})$$

is a solution of the system, where λ and $\vec{x}$ are eigenvalue eigenvector pair of matrix A.

Suppose $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are solutions of the system, then

$$\vec{y}_1'(t) = A\vec{y}_1(t) \quad \text{and} \quad \vec{y}_2'(t) = A\vec{y}_2(t)$$

Let $\vec{W} = \alpha_1 \vec{y}_1(t) + \alpha_2 \vec{y}_2(t)$, α_1, α_2 are constants.

$$\frac{d}{dt}(\vec{W}) = \alpha_1 \vec{y}_1'(t) + \alpha_2 \vec{y}_2'(t)$$

$$= \alpha_1 A\vec{y}_1(t) + \alpha_2 A\vec{y}_2(t)$$

$$= A\left(\alpha_1 \vec{y}_1(t) + \alpha_2 \vec{y}_2(t)\right)$$

$$\frac{d}{dt}(\vec{W}) = A\vec{W}$$

$\Rightarrow$ $\vec{W}$ which is a linear combination of $\vec{y}_1(t)$ and $\vec{y}_2(t)$ also satisfies the system.

In general, if $\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)$ are solutions of the system, then a linear combination

$$\alpha_1 \vec{y}_1(t) + \alpha_2 \vec{y}_2(t) + \cdots + \alpha_n \vec{y}_n(t)$$

is also a solution of the system.

This is called the ~~prin~~ superposition principle.

Homogeneous system of D.E.

$$\vec{Y}'(t) = A\,Y(t)$$

~~AAA~~ non homogeneous system

$$\vec{Y}'(t) = A\,Y(t) + \vec{g}(t)$$

Consider the system

$$\vec{Y}'(t) = A\vec{Y}(t)$$

where A is an $n \times n$ matrix.

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with corresponding eigenvectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$

Then $\vec{y}_1(t) = e^{\lambda_1 t}\vec{X}_1$, $\vec{y}_2(t) = e^{\lambda_2 t}\vec{X}_2$, $\ldots$, $\vec{y}_n(t) = e^{\lambda_n t}\vec{X}_n$

are solutions of the system.

By superposition principle,

$$\vec{Y}(t) = \alpha_1 e^{\lambda_1 t}\vec{X}_1 + \alpha_2 e^{\lambda_2 t}\vec{X}_2 + \cdots + \alpha_n e^{\lambda_n t}\vec{X}_n$$

This is the general form of the solution of the system.

$\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants.

If given an initial condition, say $\vec{Y}(0) = \vec{Y}_0$

$$\Rightarrow \quad \alpha_1 \vec{X}_1 + \alpha_2 \vec{X}_2 + \cdots + \alpha_n \vec{X}_n = \vec{Y}_0$$

This is a system of equations that can be solved to get ~~the~~ $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example: Given the system

$$\left.\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= -y_1(t) + 4y_2(t) \end{aligned}\right\} \text{——} (1)$$

(i) Find the general solution of the system.
(ii) If $y_1(0) = 1$, $y_2(0) = 2$, find the solution of the system that satisfies this conditions.

Solution

Let $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

$$\vec{Y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\vec{Y}'(t) = A\vec{Y}(t)$$

where $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

Find the eigen values and eigenvectors of A.

Let λ be an eigenvalue of A,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3.$$

For $\lambda_1 = 2$, $\quad (A - \lambda_1 I)\vec{X}_1 = \vec{0}$

$$\Rightarrow \left(\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \cancel{-x_1 + 2x_2} \qquad -x_1 + 2x_2 = 0$$

$$x_1 = 2x_2$$

take $x_2 = 1$, $x_1 = 2$

$$\vec{X}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$, $\quad (A - \lambda_2 I)\vec{X}_2 = \vec{0}$

$$\Rightarrow \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 2x_2 = 0$$

$\Rightarrow x_1 = x_2$, take $x_2 = 1$, then $x_1 = 1$

$$\vec{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We have the eigen-analysis

$$\lambda_1 = 2 \quad , \quad \vec{X}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \quad , \quad \vec{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution of the system is of the form

$$\vec{Y}(t) = \alpha_1 e^{\lambda_1 t} \vec{X}_1 + \alpha_2 e^{\lambda_2 t} \vec{X}_2$$

$$\vec{Y}(t) = \alpha_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{--- } (2)$$

We have $y_1(0) = 1$, $y_2(0) = 2$

$$\vec{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let us apply this initial condition to (2)

At $t=0$

$$\vec{Y}(0) = \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ we have the non homogeneous system

$$\alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

this gives

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow \quad \alpha_2 = 3$

$2\alpha_1 + \alpha_2 = 1$, $\Rightarrow 2\alpha_1 = 1 - \alpha_2 = -2$

$\Rightarrow \alpha_1 = -1$

$$\therefore \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore \vec{Y}(t) = (-1) e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

~~In form~~

In component form,

$$y_1(t) = -2e^{2t} + 3e^{3t}$$

$$y_2(t) = -1e^{2t} + 3e^{3t}$$

Check!!

$$y_1'(t) = -4e^{2t} + 9e^{3t} \quad \text{—— } (*)$$

$$y_1'(t) = y_1(t) + 2y_2(t)$$

$$= -2e^{2t} + 3e^{3t} + 2\left(-1e^{2t} + 3e^{3t}\right)$$

$$y_1'(t) = -4e^{2t} + 9e^{3t}. \quad \text{—— } (**)$$

Since $(*) = (**)$ we are good!

We need to check the solutions satisfy

$y_2'(t) = -y_1(t) + 4y_2(t)$ also.