Last day

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$

- Gaussian elimination to reduce the system
  - echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 5 & 4 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \checkmark$$

  - reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Example:

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 1 & 2 & -1 & -5 & -18 \\ 0 & 3 & 0 & -3 & -6 \\ -1 & 4 & 1 & 1 & 14 \end{array}\right)$$

- echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right) \quad A$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Continue with A to reduced echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & & \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \end{array}\right)$$

$R_1 = 2R_2 - R_1$

~~$R_1 = 2R_2 - R_1$~~

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right)$$

$R_1 = R_1 - 2R_2$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right)$$

$R_1 = R_1 + 2R_3$

$1 \quad 4$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$R_1 = R_1 + R_4$

$R_2 = R_2 + R_4$

$R_3 = R_3 - 2R_4$

$R_4 = R_4/2$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Example:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 1 & 3 & 4 & -2 & 3 \\ -2 & -6 & -4 & 5 & 5 \\ -1 & -3 & 2 & 1 & 6 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 4 & -1 & 5 \end{array}\right) \quad \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 + 2R_1 \\ R_4 = R_4 + R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 4 & -1 & 5 \end{array}\right) \quad R_2 = R_2/2$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right) \quad R_4 = R_4 - 4R_2$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_4 = R_4 + R_3$$

row 3, $x_4 = 3$

row 2, $x_3 = 2$

row 1, $x_1 + 3x_2 + 2x_3 - 2x_4 = -1$

$x_1 + 3x_2 + 4 - 6 = -1$

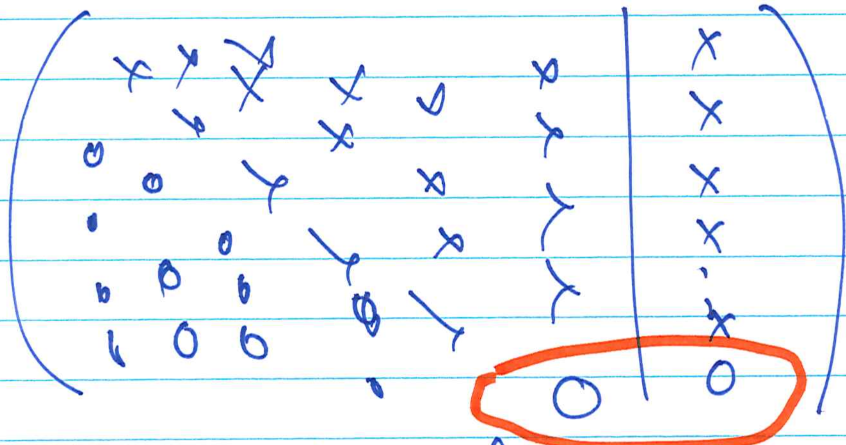
$x_1 = +1 - 3x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} +1 - 3x_2 \\ x_2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} +1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The solution of the system lies on a line with parametric form

$$\vec{x} = \begin{pmatrix} +1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 \in \mathbb{R}$$

If we have ~~then~~ an echelon form that looks like



Then the system has infinitely many solutions

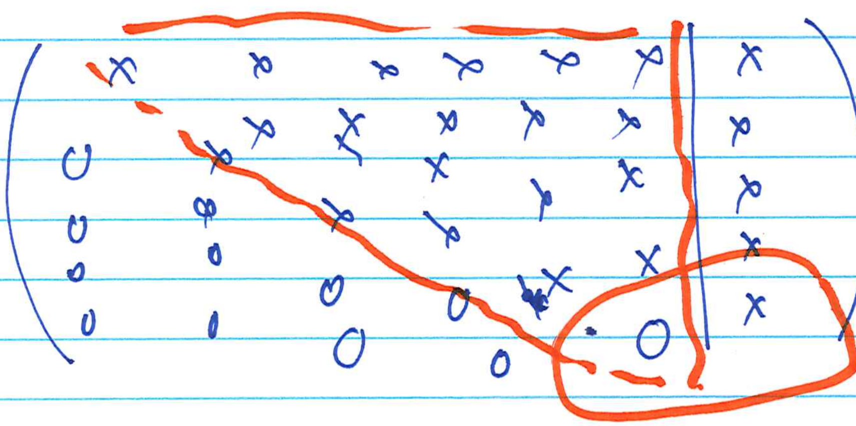
Example!

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 4 & 2 \\ -1 & -3 & 0 \\ 2 & 6 & 4 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 + R_1 \\ R_4 = R_4 - 2R_1 \end{array}$$

from row 4, $0 \cdot x + 0 \cdot y = 2$, not possible!!

$\therefore$ the system has no solution!



If we have an echelon form that looks like this, then the system has no solution.

Rank of a matrix

The rank of a matrix is the number of non-zero rows of the echelon form of the matrix.

Example!

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{array}\right) \qquad \text{rank} = 4$$

the coefficient matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, rank = 2.

We say a system has no solution if the rank of the augmented matrix is greater than the rank of the coefficient matrix.

Example:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 4 & 15 \end{array}\right), \quad \text{rank} = 3$$

rank of coefficient matrix = 3

A system has a unique solution if the rank of its augmented matrix equals the rank of the coefficient matrix and equals the number of unknowns in the system.

Example:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A system has infinitely many solution if the rank of the coefficient matrix and augmented matrix are equal (say r) and less than the number of unknowns in the system (say n)

i.e if $n > r$

and the system has $(n-r)$ parameters.

HOMOGENEOUS SYSTEMS

Consider

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n &= c_1 \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n &= c_2 \\ \vdots \qquad \vdots \qquad & \qquad \vdots \\ b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n &= c_m \end{aligned}$$

If $c_1 = c_2 = \cdots = c_m = 0$. Then the system is homogeneous. That is, if the right hand side of all the equations in the system are zero.

Solutions of homogeneous system.

* trivial solution $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *

* ~~Nontrivial Solution~~ An homogeneous system has a nontrivial solution if after reducing the system to echelon form, the number of unknown is greater than the number of non-zero rows in the matrix,

that is, if the rank of the matrix is less than the number of unknowns in the system.

Example!

$$x_1 + 2x_2 + 0 \cdot x_3 - x_4 = 0$$
$$-x_1 - 3x_2 + 4x_3 + 5x_4 = 0$$
$$x_1 + 4x_2 - 8x_3 - 9x_4 = 0$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -1 & -3 & 4 & 5 & 0 \\ +1 & 4 & -8 & -9 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 4 & 4 & 0 \\ 0 & 2 & -8 & -8 & 0 \end{array}\right) \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) R_3 = R_3 + 2R_2$$

the rank $= 2$

number of unknowns $= 4$

$\therefore$ the system has infinitely many solutions.

from row 2,

$$-x_2 + 4x_3 + 4x_4 = 0$$

$$x_2 = 4x_3 + 4x_4$$

from row 1, $x_1 + 2x_2 - x_4 = 0$

$$x_1 = x_4 - 2x_2$$

$$= x_4 - 2(4x_3 + 4x_4)$$

$$x_1 = -8x_3 - 7x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -8x_3 - 7x_4 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

x_3 ~~and~~, $x_4 \in \mathbb{R}$

Properties of solutions of homogeneous system.

(i) A homogeneous system has either

- a unique solution which is the trivial solution
- infinitely many solutions.

(ii) the addition of two solution is also a solution of the system. i.e if

$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ are solutions, then $\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}$

is also a solution.

(iii) the scalar multiple of a solution of a system is also a solution, i.e if

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a solution then $\alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$

is also a solution

Principle of superposition

The linear combination of solutions is also a solution

i.e $\alpha_1 \vec{X} + \alpha_2 \vec{Y}$ is also a solution.