Last day

~~* computing the inverse of a matrix~~

* computing the inverse of a matrix

- given an $n \times n$ matrix A
- construct $[A \mid I]$
- use elementary row operation to reduce the matrix to $[I \mid B]$
- $A^{-1} = B$.

* If $\det(A) = 0$, then A does not have an inverse, that is A is a singular matrix.

* computing the determinant of a matrix
- Given an $n \times n$ matrix
- reduce the matrix to upper or lower triangular matrix and the product of the diagonal entries is the determinant of the matrix.

Note:

Some elementary row operations change the determinant of the matrix.

Example: Find all values of λ for which the matrix

$$A = \begin{pmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{pmatrix}$$

is not invertible.

Solution: We want to find λ for which $\det(A) = 0$.

$$\det(A) = (2-\lambda)\big(-\lambda(1-\lambda) - 3\big) - 1\big(-1(1-\lambda) - 1\big) + 0$$

$$= (2-\lambda)(\lambda^2 - \lambda - 3) - 1(\lambda - 2)$$

$$= (\lambda-2)\big(-(\lambda^2 - \lambda - 3) - 1\big)$$

$$= -(\lambda - 2)(\lambda^2 - \lambda - 2) = (2-\lambda)(\lambda+1)(\lambda-2)$$

Set $\det(A) = 0 \Rightarrow (2-\lambda)(\lambda+1)(\lambda-2) = 0$

$\lambda = 2$ and $\lambda = -1$

Example: ~~Use~~ use elementary row operations to find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

$$\det(A) = \det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

$$\sim \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -6 \end{pmatrix} \quad \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\sim -\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & -2 \end{pmatrix} \quad \left(\text{Swaped rows 2 and 3}\right)$$

$$\det(A) = -1\,(1)(-1)(-2) = \underline{\underline{-2}}$$

COMPLEX NUMBERS

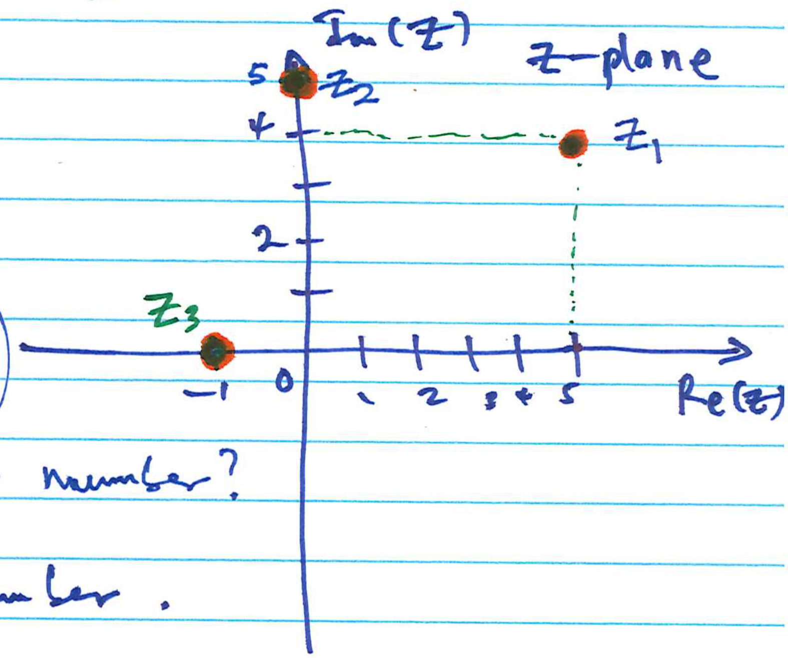
A complex number is any number of the form $z = x + iy$ where $i = \sqrt{-1}$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$ $\qquad i^2 = -1$

we say

x = real part of (z), $\mathrm{Re}(z)$

y = imaginary part of z, $\mathrm{Im}(z)$



Example:

(i) $z_1 = 5 + 4i$

(ii) $z_2 = 5i$ (purely imaginary)

Is $z_3 = -1$ a complex number?

YES! z_3 is a complex number.

In fact every real number is a complex number with ~~imaginary~~ zero imaginary part.

The set of all complex number is called the complex plane, denoted by $\mathbb{C}$.

i.e

$$\mathbb{C} = \{ z = x + iy \mid x \in \mathbb{R},\ y \in \mathbb{R} \}$$

$\Rightarrow \quad \mathbb{R} \subset \mathbb{C}$

subset

Remark: ~~equal~~ equality of complex numbers

Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$.

$z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$

i.e if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$

$\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

Addition of complex numbers

Let $z_1, z_2 \in \mathbb{C}$

Then $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

we can this in vector form as

$$(z_1 + z_2) = \Big((x_1 + x_2), (y_1 + y_2) \Big)$$

Example: $z_1 = 4+3i$, $z_2 = 7-2i$

$z_1 + z_2 = (4+7) + i(3-2)$

$= 11 + i$

Multiplication of complex numbers

Let $z_1, z_2 \in \mathbb{C}$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2$

but $i^2 = -1$

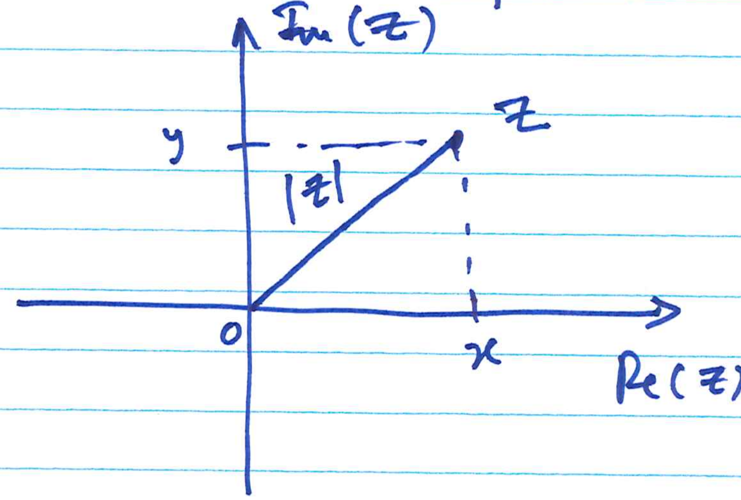
$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

Modulus of a complex number

The modulus of a complex number is the distance of the number from the origin.

Let $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$



Remark

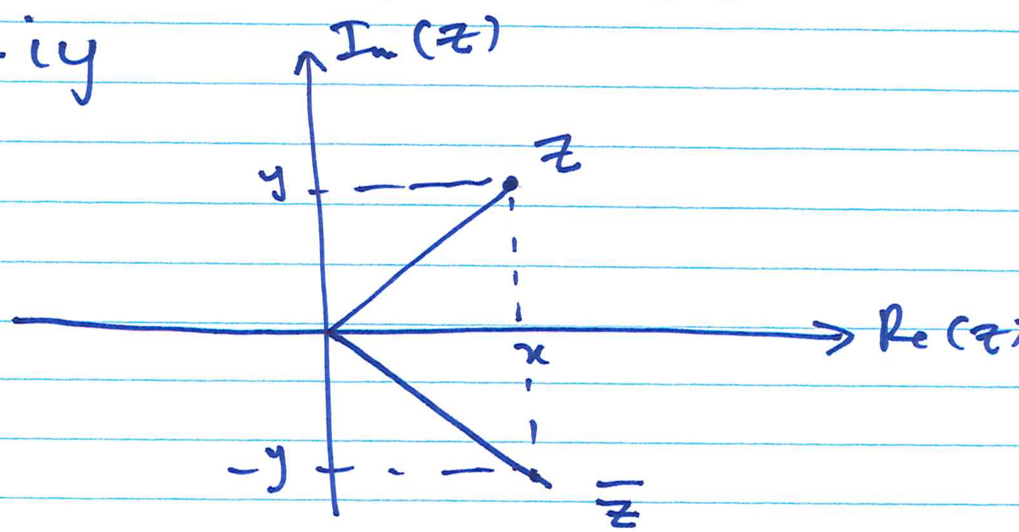
- Let z_1 ~~and~~, $z_2 \in \mathbb{C}$

$$|z_1 z_2| = |z_1||z_2|$$

~~Com~~ ~~Conjugat~~ **complex conjugate**

let $z = x + iy$, the complex conjugate of z is

$$\bar{z} = x - iy$$



~~Con~~

properties of complex conjugate

Let $z_1, z_2 \in \mathbb{C}$

(i) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}$

(iv) $z_1 \overline{z_1} = |z_1|^2$

(v) $\dfrac{z_1}{z_2} = \dfrac{z_1 \times \overline{z_2}}{z_2 \times \overline{z_2}} = \dfrac{z_1 \overline{z_2}}{|z_2|^2}$

Example: Let $z_1 = 4+3i$, $z_2 = 2+4i$

Find z_1/z_2, ~~comp~~ put your result in the form $x+iy$.

$\dfrac{z_1 \times \overline{z_2}}{z_2 \times \overline{z_2}} = \dfrac{z_1 \overline{z_2}}{|z_2|^2} = \dfrac{(4+3i)(2-4i)}{20} = \dfrac{20-10i}{20}$

$= 1 - i/2$

Let $z = i4 + 5$

$\bar{z} = 5 - 4i$

Let $z = x + iy$, $\bar{z} = x - iy$

- $z + \bar{z} = (x+iy) + (x-iy)$
$= 2x$

$\Rightarrow x = \frac{1}{2}(z + \bar{z}) \Rightarrow Re(z) = \frac{1}{2}(z + \bar{z})$

- $z - \bar{z} = i2y$

$\Rightarrow y = Im(z) = \frac{1}{2i}(z - \bar{z})$.

Example! Find the root of the ~~tow~~ equation

$x^2 + 2 = 0$, $x \in \mathbb{C}$

$x = \pm\sqrt{-2} = \pm\sqrt{-1 \times 2} = \pm\sqrt{-1}(\sqrt{2})$

$x = \pm i\sqrt{2}$

Theorem

If we are working in complex ~~co~~plane, every polynomial can be factorized completely.

COMPLEX MATRICES

An $n \times m$ matrix is a complex matrix if the entries are complex numbers.

Examples:

$$A = \begin{pmatrix} 2+4i & 3+i & 4+3i \\ 4 & 7+2i & 5 \\ 2i & 3i & 1+i \end{pmatrix}$$

is a complex matrix.

Determinant of complex matrices

$-16i(1+i)$

$i+i^2$

$-1+i$

Example!

Let $A = \begin{pmatrix} 2+4i & 0 & 4i \\ 2+3i & 1+i & 4+i \\ 4 & 0 & 2 \end{pmatrix}$

$$\det(A) = (2+4i)\big(2(1+i)-0\big) - 0 + 4i\big(0-4(1+i)\big)$$

$$= (2+4i)(2+2i) + 4i(-4-4i)$$

$$= 4(1+2i)(1+i) - 16(-1+i)$$

$$= 4(1+i+2i+i^2 2) - 16(-1+i)$$

$$= 4(-1+3i) - 16(-1+i)$$

$$= 4(-1+3i) + 16(1-i)$$

$$= 12-4i$$

$\det(A) = 12-4i$

Since $\det(A) \neq 0$, the inverse exists.

HOMOGENEOUS COMPLEX LINEAR SYSTEMS

This is a homogeneous system of equation with complex coefficients.

Example: Solve the following homogeneous system.

$$4i\,x_1 + \quad + 12x_3 = 0$$

$$x_1 + i x_2 + - x_3 = 0$$

$$2x_2 + (6+2i)\,x_3 = 0$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 4i & 0 & 12 & 0 \\ 1 & i & -1 & 0 \\ 0 & 2 & (6+2i) & 0 \end{array}\right)$$

$$\frac{12}{4i} = \frac{3}{i}\frac{\times i}{\times i} = -3i$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 1 & i & -1 & 0 \\ 0 & 2 & (6+2i) & \end{array}\right) \quad R_1 = R_1/4i$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & i & -1+3i & 0 \\ 0 & 2 & (6+2i) & 0 \end{array}\right) \quad R_2 = R_2 - R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & 1 & 3+i & 0 \\ 0 & 1 & (3+i) & 0 \end{array}\right) \quad \begin{array}{l} R_2/i \\ R_3/2 \end{array}$$

$$\frac{-1+3i}{i} \frac{\times i}{\times i}$$

$$\frac{(-1+3i)\,i}{-1}$$

$$(1-3i)\,i$$

$$i - 3i^2$$

$$3+i$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -3i & 0 \\ 0 & 1 & 3+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 = R_3 - R_2$$

from row 2, $\quad x_2 + (3+i)x_3 = 0$

$$x_2 = -(3+i)\,x_3$$

from row 1, $\quad \cancel{x_1 - 3i x_2 = 0 \Rightarrow ?}$

$$x_1 - 3i x_3 = 0 \quad \Rightarrow \quad x_1 = +3i x_3 .$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} +3i \\ -(3+i) \\ 1 \end{pmatrix}, \quad t \text{ constant.}$$