Part A - Short Answer Questions, 1 mark each

For Questions A1-A3 below, let

$$\mathbf{a} = [1, 1, 5]$$
$$\mathbf{b} = [2, -2, 1]$$

The answers for questions A1-A2 should be in the form $[x, y, z]$ with x, y, and z determined.

A1: Compute $\mathbf{a} \times \mathbf{b}$

$$\underline{a} \times \underline{b} = \text{"det"} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 5 \\ 2 & -2 & 1 \end{pmatrix} = (11, 9, -4)$$

A2: Compute $\text{proj}_{\mathbf{a}}\mathbf{b}$

$\underline{a} \cdot \underline{b} = 5$

$\|\underline{a}\|^2 = 27$

$$\text{proj}_{\underline{a}} b = \frac{5}{27}(1, 1, 5) = \left(\frac{5}{27}, \frac{5}{27}, \frac{25}{27}\right)$$

A3: Determine the value of $\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
Note: do not try to determine the value of θ.

$\|b\| = 3$

$$\cos\theta = \frac{5}{3\sqrt{27}} = \frac{5}{9\sqrt{3}}$$

A4: Give two vectors of length 10 that are parallel to $[3, 1, 5]$.

$$\|(3,1,5)\| = \sqrt{9+1+25} = \sqrt{35} \qquad \pm \frac{10}{\sqrt{35}}(3, 1, 5).$$

A5: Consider the following lines of MATLAB code:

```
A = zeros(3,2);
for i=1:2
   A(i,i) = i
end
```

What is the result in A?

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

A6: Consider a linear system of eight equations in six unknowns. Circle below *all possible* forms of the solution set to the system.

(a) The system might have no solutions
(b) The system might have exactly one solution
(c) The system might have exactly two solutions
(d) The system might have an infinite number of solutions
(e) The set of solutions might be three dimensional

A7: Determine whether the lines L_1 and L_2 below intersect. If they do, find the point of intersection. If they do not, justify your answer.

$$L_1 \; : \; \mathbf{x} = [3,0,2] + s[-1,1,2]$$
$$L_2 \; : \; \mathbf{x} = [0,0,7] + t[1,2,-1]$$

component 1: $3 - s = t$ } $\Rightarrow t = 1, \; s = 2.$
2: $s = 2t$

3: $2 + 2(2) = 7 - 1$?
$6 = 6$ ✓

Yes, intersect at $(3,0,2) + 2(-1,1,2) = (1,2,6)$.

A8: Find the distance from the origin to the plane $3x - y - 4z = -2$.

Consider the line L: $s(3,-1,-4)$.
L intersects the plane when
$3(3s) - (-s) - 4(-4s) = -2$
$\Rightarrow s = -1/13$, point $\left(\frac{-3}{13}, +\frac{1}{13}, -\frac{4}{13}\right)$.
The distance desired is the distance of the origin to this point.

$\left\| \left(-\frac{3}{13}, \frac{1}{13}, \frac{4}{3}\right) \right\| = \frac{\sqrt{26}}{13} = \sqrt{\frac{2}{13}}$

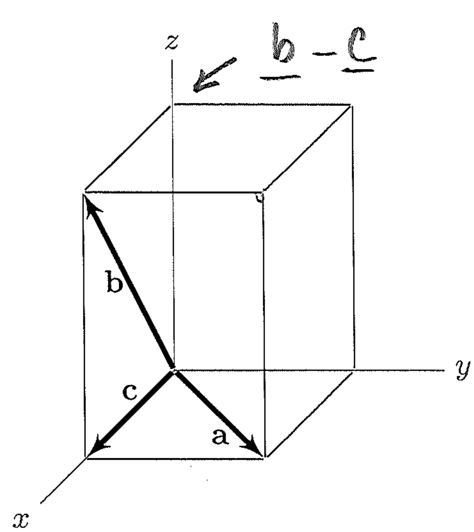
A9: Below is an augmented matrix representing a system of linear equations. Describe all solutions to the system of equations in parametric form.

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 4 & 3 & -1 & 3 \\ 0 & 1 & 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$x_6 = t$, $x_5 = s$, $x_4 = -1$, $x_3 = u$, $x_2 = 4-2s$
$x_1 = 7 - 3s + t$.

$\underline{x} = (7-3s+t, 4-2s, u, -1, s, t)$.

A10: Below is a picture of a rectangular box in $\mathbb{R}^3$. The farthest corner from the origin is (p_1, p_2, p_3). Write $(0, 0, p_3)$ as a linear combination of the named vectors in the picture.



Test1 B

B1 (Solution)

Consider the linear system below for the unknowns x,y, and z. It is known that the system has a unique solution.

$$\begin{aligned} 2x - 2y + 6z &= 4 \\ -2x + 3y - z &= 5 \\ 6x - 5y + 3z &= 1 \end{aligned}$$

(a) (1 mark) write the system in augmented matrix.

$$\begin{pmatrix} 2 & -2 & 6 & \vdots & 4 \\ -2 & 3 & -1 & \vdots & 5 \\ 6 & -5 & 3 & \vdots & 1 \end{pmatrix}$$

(b) (3 mark) Do row operations (Gaussian elimination) on the augmented matrix to change it to echelon form.

$$\begin{pmatrix} 2 & -2 & 6 & \vdots & 4 \\ -2 & 3 & -1 & \vdots & 5 \\ 6 & -5 & 3 & \vdots & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ -2 & 3 & -1 & \vdots & 5 \\ 6 & -5 & 3 & \vdots & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 1 & 5 & \vdots & 9 \\ 0 & 1 & -15 & \vdots & -11 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 1 & 5 & \vdots & 9 \\ 0 & 0 & -20 & \vdots & -20 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 1 & 5 & \vdots & 9 \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

(c) (1 mark) Find the solution to the problem from the form above.

From row 3, $z = 1$.

From row 2, $y + 5z = 9$ and this gives $y = 4$.

From row 1, $x - y + 3z = 2$, $x = 3$.

Therefore, the solution to the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

B2: For each of the systems described below as augmented matrices that have been put into reduced row echelon form, determine whether the systems have

- no solutions
- a single solution (in this case, find the solution)
- an infinite number of solutions (in this case, find a parametric description of all solutions).

Each part is worth 1 mark.

(a) $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 0 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(e) $\left[\begin{array}{cccc|c} 1 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$

(a) no solutions

(b) a single solution:
$(5, -2, 7)$

(c) an infinite number of solutions:
$(1-5t, t, 2)$, or $(t, (1-t)/5, 2)$, or equivalent, $t \in \mathbb{R}$

(d) an infinite number of solutions:
$(3-3t, t, 2)$, or $(t, (3-t)/3, 2)$, or equivalent, $t \in \mathbb{R}$

(e) an infinite number of solutions:
$(1-5t, t, -2, 10)$, $(t, (1-t)/5, -2, 10)$, or equivalent, $t \in \mathbb{R}$

B3: Given the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}.$$

(a) [1 mark] Show that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent.

(b) [2] Find all possible combinations of scalars $\{x_1,\ x_2,\ x_3\}$ such that

$$x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{v}.$$

(c) [2] Let S_1 be the plane that contains vectors $\mathbf{a}$ and $\mathbf{b}$ and the origin; and S_2 be the plane that contains vectors $\mathbf{c}$ and $\mathbf{v}$ and the origin. Find the parametric equation of the line of intersection between S_1 and S_2.

(a)

$$\det(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 3 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} = 1 - (-8) = 9 \neq 0,$$

therefore $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent.

(b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 2 & -1 & 2 & 2 \\ 3 & 0 & -1 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & -3 & 2 & -10 \\ 0 & -3 & -1 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & -3 & 2 & -10 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$(x_1, x_2, x_3) = (\frac{28}{9}, \frac{26}{9}, -\frac{2}{3})$.

(c)

$$\begin{aligned} \operatorname{span}\{\mathbf{a},\mathbf{b}\} \cap \operatorname{span}\{\mathbf{c},\mathbf{v}\} &= \operatorname{span}\{\mathbf{a},\mathbf{b}\} \cap \operatorname{span}\{\mathbf{c}, x_1\mathbf{a} + x_2\mathbf{b}\} \\ &= \operatorname{span}\{\mathbf{a},\mathbf{b}\} \cap \operatorname{span}\{x_1\mathbf{a} + x_2\mathbf{b}\} \\ &= \operatorname{span}\{x_1\mathbf{a} + x_2\mathbf{b}\} \\ &= \operatorname{span}\left\{\frac{28}{9}(1,2,3)^T + \frac{26}{9}(1,-1,0)^T\right\} \\ &= \operatorname{span}\left\{(6, \frac{10}{3}, \frac{28}{3})\right\}. \end{aligned}$$

So the parametric equation is

$$x = 9t, \quad y = 5t, \quad z = 14t, \quad t \in \mathbb{R}.$$