Part A - Short Answer Questions, 1 mark each

A1: Compute the determinant of the matrix

$$\det \begin{bmatrix} 10 & 19 & 8 & 53 \\ 0 & 1 & 19 & 2 \\ 0 & 0 & \frac{1}{5} & 4 \\ 0 & 0 & 0 & 7 \end{bmatrix} = 14.$$

A2: Given that $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, and $\mathbf{x} = [2\ 1]$, calculate $(\mathbf{x}A)^T$.

$$\underset{\sim}{x}A = [4\ 1\ 1] \quad , \quad (\underline{x}A)^T = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

A3: For $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$, find A^{-1}.

$$A^{-1} = \begin{bmatrix} \frac{4}{10} & \frac{-3}{10} \\ \frac{2}{10} & \frac{1}{10} \end{bmatrix}$$

A4: Which of the following are true for *all* 4×4 matrices A, with $\det(A) = 10$? Circle all that apply.

- **(a)** The reduced row echelon form of A is the 4×4 identity matrix.
- **(b)** A is invertible.
- **(c)** The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
- **(d)** The rank of A is 10.
- **(e)** $A = A^T$.

A5: Consider the following lines of MATLAB code:

```
A = [1 1; 1 1];
rref(A)
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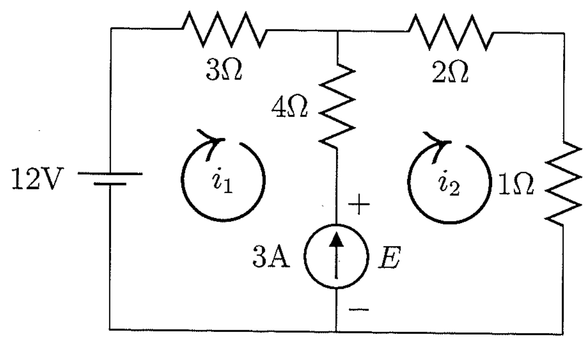
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

What is the result of the last line above?

Questions A6-A7 concern the resistor network below.



A6: Write the linear equation involving the unknown loop currents and current source voltage that corresponds to summing the voltage drops around loop 1 (Kirchhoff's second law).

$7i_1 - 4i_2 + E = 12.$

A7: Write the linear equation that matches the loop currents to the current source.

$-i_1 + i_2 = 3.$

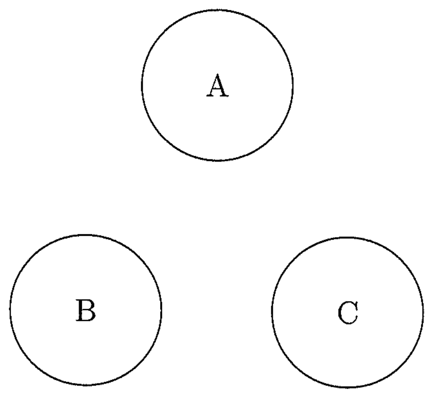
A8: Suppose the matrix A below satisfies $A^T = A^{-1}$. What are all possible values of c and d?

$$A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

$A^TA = I \qquad \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = \begin{bmatrix} c^2 & cd \\ cd & 1+d^2 \end{bmatrix}.$

$d=0, \quad c=\pm 1.$

Questions A9-A10 below concern a game described as follows: You place a game piece on one of the three circles below. You roll a 6-sided die to figure out where your game piece moves. If you roll 1, you stay in your space. If you roll 2 or 3, you move your piece clockwise. If you roll 4, 5, or 6, you move your piece counterclockwise.



A9: Write a probability transition matrix P for the game. Use the ordering ABC for the states.

$$\mathbb{P} = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}.$$

A10: Suppose you start on Circle A (before you've rolled the die at all). What is the probability that after two rolls, your piece is in Circle A?

$$\underline{X_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{X_1} = \mathbb{P}\,\underline{X_0} = \begin{bmatrix} 1/6 \\ 1/2 \\ 1/3 \end{bmatrix}. \quad \underline{X_2} = \mathbb{P}\,\underline{X_1} = \begin{bmatrix} \frac{1}{36} + \frac{1}{6} + \frac{1}{6} \\ ? \\ ? \end{bmatrix} \leftarrow \text{don't care.}$$

probability $\frac{13}{36}$.

Part B - Long Answer Questions, 5 marks each

B1: Match the matrices A below with their inverse. One mark each.

(a) $A = \begin{bmatrix} -2 & -3 & -1 \\ -1 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$ (B).

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (D)

(c) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (A)

(d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ (A)

(e) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (C)

(A) A^{-1} does not exist.

(B) $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & -1 \\ 3 & -5 & 1 \end{bmatrix}$

(C) $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

(D) $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$

(E) $A^{-1} = \begin{bmatrix} 4/3 & -1/2 \\ 1/3 & 0 \\ -2/3 & 1/2 \end{bmatrix}$

(F) A^{-1} is not in the list above.

B2: Let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Given that

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\2\end{bmatrix},$$

- **(a)** [1 mark] Write the vector [1 0 0] as a linear combination of [1 1 1] and [0 1 1].
- **(b)** [3] Let B be the matrix representation of T. Find B.
- **(c)** [1] Determine whether B is invertible using a determinant calculation.

Test2 A

B2 (Solution)

(a) [1 mark] Write the vector [1 0 0] as a linear combination of [1 1 1] and [0 1 1].

We want to find scalars α_1 and β_1 such that

$$\alpha_1\ [\,1\ 1\ 1\,] + \beta_1\ [\,0\ 1\ 1\,] = [\,1\ 0\ 0\,]$$

By inspection, we notice that $\alpha_1 = 1$ and $\beta_1 = -1$ works.

(b) [3 marks] Let B be the matrix representation of T. Find B.

If B is the matrix representation of the transformation T, then

$$B = \begin{pmatrix} \vdots & \vdots & \vdots \\ T(\overrightarrow{e}_1) & T(\overrightarrow{e}_2) & T(\overrightarrow{e}_3) \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where $\overrightarrow{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\overrightarrow{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\overrightarrow{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are the standard basis vectors in three dimension.

Now, let us find the columns of matrix B. From (a), we already have

$$\overrightarrow{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_1 \begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \text{where} \quad \alpha_1 = 1,\ \beta_1 = -1.$$

Therefore,

$$T(\overrightarrow{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = T\left(\alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_1 \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$$

Since the transformation is linear, we have

$$T(\overrightarrow{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \alpha_1 T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + \beta_1 T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \alpha_1 \begin{bmatrix}1\\0\\-1\end{bmatrix} + \beta_1 \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}.$$

For the second column, we can also write $\overrightarrow{e}_2$ as

$$\overrightarrow{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = \alpha_2 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_2 \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \text{where} \quad \alpha_2 = 1,\ \beta_2 = -1.$$

Therefore,

$$T(\overrightarrow{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\alpha_2 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_2 \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

Since the transformation is linear, we have

$$T(\overrightarrow{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \alpha_2 T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + \beta_2 T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \alpha_2 \begin{bmatrix}1\\0\\-1\end{bmatrix} + \beta_2 \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\-3\end{bmatrix}.$$

Similarly, for the third column, we write

$$\overrightarrow{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \alpha_3 \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_3 \begin{bmatrix}1\\0\\1\end{bmatrix} + \gamma_3 \begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \text{where} \quad \alpha_3 = -1,\ \beta_3 = 1,\ \&\ \gamma_3 = 1.$$

Therefore,

$$T(\overrightarrow{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\alpha_3\begin{bmatrix}1\\1\\1\end{bmatrix} + \beta_3\begin{bmatrix}1\\0\\1\end{bmatrix} + \gamma_3\begin{bmatrix}0\\1\\1\end{bmatrix}\quad\right)$$

Since the transformation is linear, we have

$$T(\overrightarrow{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \alpha_3\,T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + \beta_3\,T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) + \gamma_3\,T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \alpha_3\begin{bmatrix}1\\0\\-1\end{bmatrix} + \beta_3\begin{bmatrix}1\\-1\\2\end{bmatrix} + \gamma_3\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$T(\overrightarrow{e}_3) = (-1)\times\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\\2\end{bmatrix} + \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\-2\\3\end{bmatrix}.$$

Thus,

$$B = \begin{pmatrix}\vdots & \vdots & \vdots\\ T(\overrightarrow{e}_1) & T(\overrightarrow{e}_2) & T(\overrightarrow{e}_3)\\ \vdots & \vdots & \vdots\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 1 & 1 & -2\\ -1 & -3 & 3\end{pmatrix}.$$

(c) [1 marks] Determine whether B is invertible using determinant calculation.

$$\mathrm{B} = \begin{vmatrix}0 & 0 & 1\\ 1 & 1 & -2\\ -1 & -3 & 3\end{vmatrix} = -2$$

Since $\det(B) \neq 0$, B is invertible.

B3: Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector across the line $y = x$, and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector $45°$ counter-clockwise.

(a) [1 mark] Write a matrix R_1 such that $T_1(\mathbf{x}) = R_1\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$.

(b) [1] Write a matrix R_2 such that $T_2(\mathbf{x}) = R_2\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$.

(c) [1] Suppose $T_3 = T_1 \circ T_2$. That is, T_3 is the transformation that takes an input vector in $\mathbb{R}^2$, rotates it counter-clockwise, then reflects it. Find a matrix R_3 such that $T_3(\mathbf{x}) = R_3\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$.

(d) [2] Find a nonzero vector $\mathbf{x}$ in $\mathbb{R}^2$ such that $T_1(\mathbf{x}) = T_2(\mathbf{x})$, or show with a calculation that none exists.

Midterm Exam II question B3, Version A

Answers:

(a) $R_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $R_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $R_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(d) The coordinates of such vector would satisfy the following system of equations:

$$\begin{aligned} x_2 &= \tfrac{1}{\sqrt{2}}x_1 - \tfrac{1}{\sqrt{2}}x_2, \\ x_1 &= \tfrac{1}{\sqrt{2}}x_1 + \tfrac{1}{\sqrt{2}}x_2. \end{aligned}$$

From here, by adding the two equations together, we get the equation $x_1 + x_2 = \sqrt{2}x_1$. For the set of solutions we can choose x_1 or x_2 as a free variable. If we set $x_1 = s$ then $x_2 = (\sqrt{2}-1)s$.

One possible vector is $\vec{x} = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$.

If we set $x_2 = t$ then $x_1 = (\sqrt{2}+1)t$, so $\vec{x} = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ is another possible solution.