(i) Initial Value problem (IVP)

$$y'' + by' + cy = f(t) \quad , \quad t > t_0$$

$$\left.\begin{aligned} y(t_0) &= \alpha \\ y'(t_0) &= \beta \end{aligned}\right\} \text{ initial conditions}$$

* The solutions of this equation is valid for all $t > t_0$

(ii) Boundary Value Problem (BVP)

$$y'' + by' + cy = f(x) \quad , \quad x \in [a, b]$$

$$\left.\begin{aligned} y(a) &= \alpha \\ y(b) &= \beta \end{aligned}\right\} \text{ the boundary conditions.}$$

* The solution is valid only in the interval $[a, b]$.

Resonance

We started with

$$y'' + y = \cos(t) \qquad (1)$$

For the homogeneous solution, $y'' + y = 0$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

$$\therefore y_H = c_1 \cos(t) + c_2 \sin(t).$$

For y_p,

Guess; $y_p = At\cos(t) + Bt\sin(t)$.

$$y_p' = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)$$

$$y_p'' = (-2A - Bt)\sin(t) + (-At + 2B)\cos(t)$$

put y_p into (1),

$$(-2A \cancel{-Bt})\sin(t) + (\cancel{-At} + 2B)\cos(t)$$
$$+ \cancel{At\cos(t)} + \cancel{Bt\sin(t)} = \cos(t)$$

$$-2A\sin(t) + 2B\cos(t) = \cos(t)$$

$$\Rightarrow 2B = 1 \Rightarrow B = \tfrac{1}{2}$$

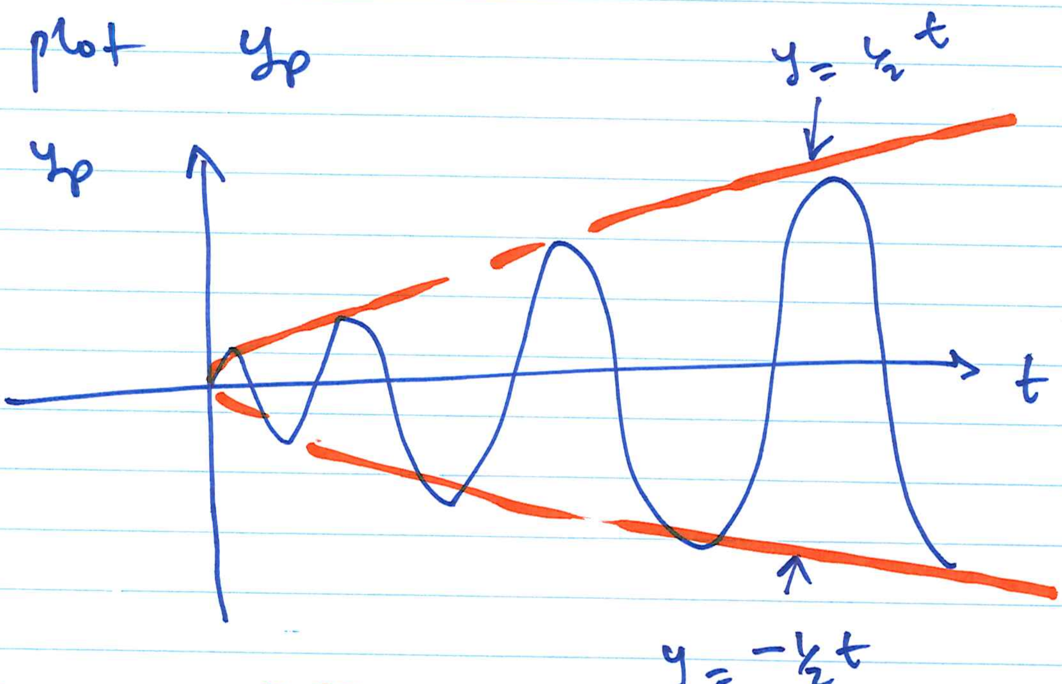
Collecting coefficients of $\sin(t)$, we have

$A = 0$.

$$\therefore \ y_p = \frac{1}{2} t \sin(t)$$

$$y(t) = C_1 \cos(t) + C_2 \sin(t) + \tfrac{1}{2} t \sin(t).$$

y_p plot



** new **

This shows that if we force a system at the same frequency as its natural frequency, it will lead to an oscillation whose amplitude increases over time.

Beats

Consider an undamped oscillator

$$my'' + ky = \cos(\omega t) \quad \text{——} \quad (1)$$

$$y'' + \frac{k}{m} y = \frac{1}{m}\cos(\omega t)$$

where $\omega_0 = \sqrt{k/m} \neq \omega$

$$y'' + \omega_0^2 y = \frac{1}{m}\cos(\omega t)$$

For y_H,

$$y_H(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

For the particular solution,

Guess: $y_p = A\cos(\omega t) + B\sin(\omega t)$

Sub. y_p into (1),

$$-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) + \omega_0^2 \big(A\cos(\omega t) + B\sin(\omega t)\big) = \frac{1}{m}\cos(\omega t)$$

Let us collect coefficients;

For $\cos(\omega t)$, we have

$$-\omega^2 A + \omega_0^2 A = \frac{1}{m}$$

$$\Rightarrow \quad A = \frac{1}{m(\omega_0^2 - \omega^2)}$$

For $\sin(\omega t)$,

$$-\omega^2 B + \omega_0^2 B = 0$$

$$\Rightarrow B = 0 \quad (\text{Since } \omega \neq \omega_0)$$

$$\therefore y_p = \frac{1}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$\therefore y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{1}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

The initial conditions are

$$y(0) = 0, \quad y'(0) = 0$$

Applying $y(0) = 0$,

$$\Rightarrow \quad C_1 + \frac{1}{m(\omega_0^2 - \omega^2)} = 0$$

$$\Rightarrow \quad C_1 = -\frac{1}{m(\omega_0^2 - \omega^2)}$$

$$y' = -\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t) - \frac{\omega \sin(\omega t)}{m(\omega_0^2 - \omega^2)}$$

$y'(0) = 0$

$$\Rightarrow \quad C_2 = 0 .$$

$$\therefore y(t) = -\frac{1}{m(\omega_0^2 - \omega^2)} \cos(\omega_0 t) + \frac{1}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$y = \frac{1}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t) \right) \quad \text{— } *1$$

Let us write $y(t)$ as a product of sine functions.
Recall,

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B) \quad \text{— } *2$$

From (*1) and (*2)

Let $A+B = \omega t$ and $A-B = \omega_0 t$

$$\Rightarrow \quad A = \frac{1}{2}(\omega + \omega_0)t$$

$$B = \frac{1}{2}(\omega - \omega_0)t.$$

$$y = \frac{1}{m(\omega_0^2 - \omega^2)}\left[-2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right)\right]$$

$$y = \frac{-2}{m(\omega_0^2 - \omega^2)}\left[\sin\left(\frac{(\omega + \omega_0)t}{2}\right) \sin\left(\frac{(\omega - \omega_0)t}{2}\right)\right]$$

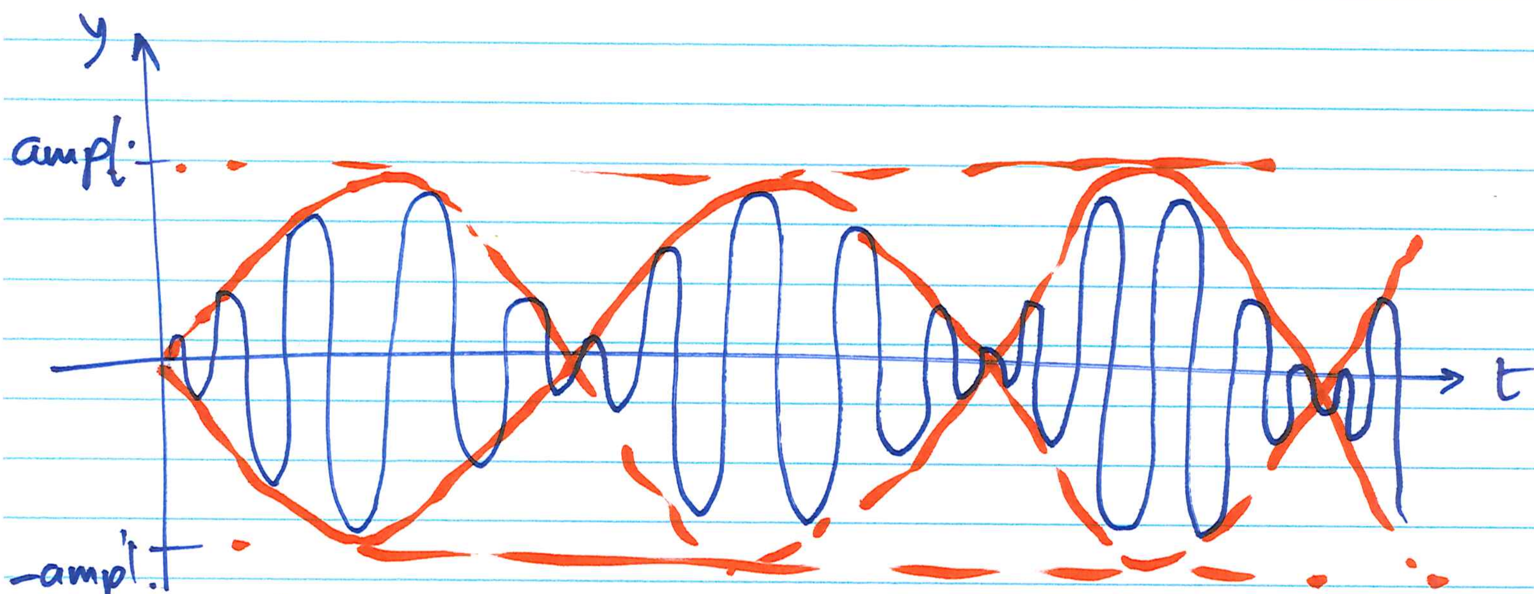
$$y = \begin{pmatrix}\text{constant} \\ \text{amplitude}\end{pmatrix} \times \begin{pmatrix}\text{higher} \\ \text{frequency} \\ \text{term}\end{pmatrix} \times \begin{pmatrix}\text{lower} \\ \text{frequency} \\ \text{term}\end{pmatrix}$$

Demo. ·—

observations:

(i) If $\omega \approx \omega_0$ $(\omega_0 \neq \omega)$

* The amplitude of the solution is larger
* The higher frequency term oscillates much faster than the lower frequency term.



for $\omega \approx \omega_0$.