Continuation from Last class ...
Details of how to compute eigenvectors, $\vec{V}_2$
We have

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $u_2 = 1$,

$$2i u_1 - 2u_2 = 0$$

$$2i u_1 = 2$$

$$u_1 = \frac{\cancel{2}}{\cancel{2}i} = \frac{1}{i} \times \frac{(-i)}{(-i)} = \frac{-i}{-i^2} = -i$$

$$\therefore \vec{V}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Remark

Given a 2×2 ~~matrix~~ real matrix with complex eigenvalue λ_1 and corresponding eigenvector $\vec{V}_1$. Then the complex conjugate $\bar{\lambda}$ is also an eigenvalue of the matrix with corresponding eigenvector $\overline{\vec{V}_1}$.

$\therefore$ the solution of the system is

$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v_1} + c_2 e^{\lambda_2 t} \vec{v_2}$$

$$= c_1 e^{(1+2i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Observe that this solution is complex-valued. It is important to put the solution in 'real form'. Let's do this!

$$\vec{Y}(t) = c_1 e^{t} \cdot e^{2it} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{t} \cdot e^{-2it} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= e^{t}\left[c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} \left(\cos(2t) + i\sin(2t)\right) + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} \left(\cos(2t) - i\sin(2t)\right) \right]$$

$$= e^{t} \begin{bmatrix} c_1 i\cos(2t) - c_1\sin(2t) - c_2 i\cos(2t) - 1c_2\sin(2t) \\ c_1\cos(2t) + ic_1\sin(2t) + c_2\cos(2t) - ic_2\sin(2t) \end{bmatrix}$$

$$= e^{t} \begin{bmatrix} \overbrace{-c_1\sin(2t) - c_2\sin(2t)}^{-(c_1+c_2)\sin(2t)} + i(c_1-c_2)\cos(2t) \\ (c_1+c_2)\cos(2t) + i(c_1-c_2)\sin(2t) \end{bmatrix}$$

Let $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$

$$\vec{Y}(t) = e^t \begin{bmatrix} -k_1 \sin(2t) + k_2 \cos(2t) \\ k_1 \cos(2t) + k_2 \sin(2t) \end{bmatrix}$$

$$\vec{Y}(t) = k_1 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + k_2 e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

This is the general solution of the system which is in "real form".

Que!
Is there a faster way of getting to the 'real' solution? YES!! ☺

This solution is the same as

$$\vec{Y}(t) = k_1 \mathrm{Re}\left(e^{\lambda_1 t} \vec{V}_1 \right) + k_2 \mathrm{Im}\left(e^{\lambda_1 t} \vec{V}_1 \right)$$

Let us try this!!

We have

$$\lambda_1 = 1+2i \quad \text{and} \quad \vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$e^{\lambda_1 t}\vec{v}_1 = e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{2it}$$

$$= e^{t}\begin{pmatrix} i \\ 1 \end{pmatrix}\left(\cos(2t) + i\sin(2t)\right)$$

$$= e^{t}\begin{bmatrix} i\cos(2t) - \sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$e^{\lambda_1 t}\vec{v}_1 = \underbrace{e^{t}\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}}_{Re\left(e^{\lambda_1 t}\vec{v}_1\right)} + i\underbrace{e^{t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}}_{Im\left(e^{\lambda_1 t}\vec{v}_1\right)}$$

$\therefore$ The general solution is

$$\vec{Y}(t) = k_1 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + k_2 e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

which is the same as what we obtained earlier.

Let us apply the initial condition $\vec{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to get k_1 and k_2.

$$\vec{Y}(0) = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow k_1 = 2, \quad k_2 = 1$

$$\therefore \vec{Y}(t) = 2 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + 1 \cdot e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

Remark

Given a system of ODE

$$\vec{y}'(t) = A\vec{y}(t)$$

where A has complex eigenvalues,

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta .$$

* the real part of the eigenvalue governs the growth / decay of the solution
* the imaginary part of the eigenvalue governs the frequency of oscillation of the solution.
* If $Re(\lambda_1) = Re(\lambda_2) > 0$, the solution ~~osc~~ oscillates and grows.
* If $Re(\lambda_1) = Re(\lambda_2) < 0$, solution oscillates and decay
* If $Re(\lambda_1) = Re(\lambda_2) = 0$, the solution is periodic.

VECTOR FIELD

Consider

$$y_1'(t) = y_1 - 2y_2$$

$$y_2'(t) = 2y_1 + y_2$$

We can write this system in the form

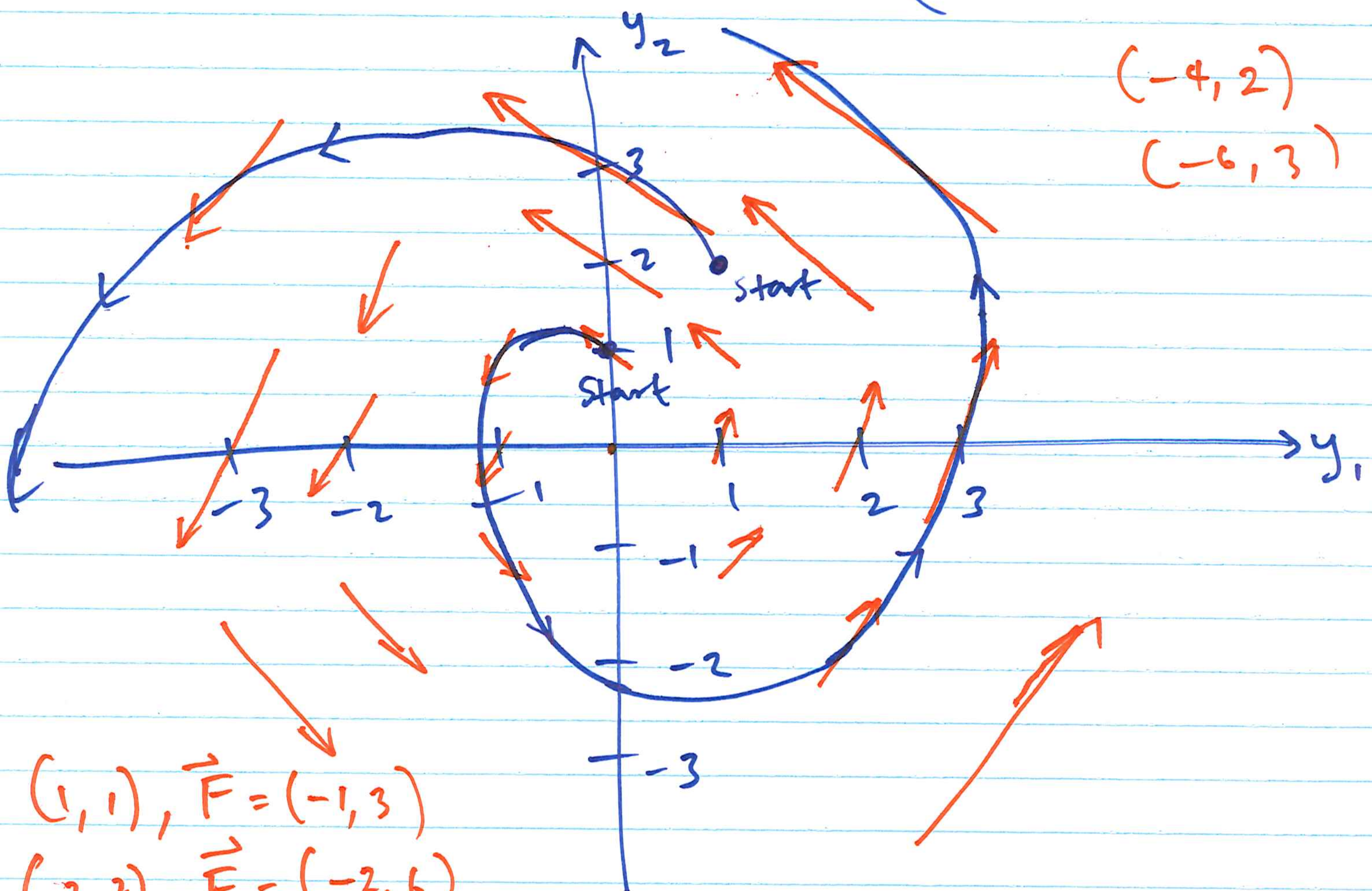
$$\vec{y}'(t) = \vec{F}(\vec{y}(t))$$

where $\vec{y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\vec{F}(\vec{y}(t)) = \begin{pmatrix} y_1 - 2y_2 \\ 2y_1 + y_2 \end{pmatrix}$

The vector-valued function $\vec{F}$ is called ~~the~~ a vector field. This function ~~tells~~ ~~us~~ gives the magnitude and direction of our solution at each point (y_1, y_2).

Let us draw the vector field for this system.

$$\vec{F}(\vec{Y}(t)) = \vec{F}((y_1, y_2)) = (y_1 - 2y_2,\ 2y_1 + y_2)$$



$(1,1)$, $\vec{F} = (-1, 3)$
$(2,2)$, $\vec{F} = (-2, 6)$

The idea is that for each point (y_1, y_2), we find the value of the vector $\vec{F}((y_1, y_2))$. The ~~vector~~ magnitude and direction of this vector $\vec{F}$ is plotting on the $y_1 y_2$-plane.