(linear analysis)

<u>Conclusion</u> (From the phase portraits and matlab simulation (numerical solution))

(i) If $X(0) > 0$, $Y(0) = 0$, then the population converges to $(3/4, 0)$

(ii) If $X(0) = 0$, $Y(0) > 0$, then the population converges to $(0, 1)$

(iii) If $X(0) > 0$ and $Y(0) > 0$, then the solution converges to $(4/7, 5/7)$.
And this implies that for any initial population $X > 0$ and $Y > 0$, the two populations will co-exist.

Summary: Linear analysis

Given a nonlinear system

$$\vec{Y}' = \vec{F}(\vec{Y}) \qquad (1)$$

(i) Find all the equilibria of the system.

(ii) Linearize the system close to each of the steady-state solutions to have

$$\vec{u}'(t) = D\vec{F}(\vec{Y}_0)\,\vec{u}$$

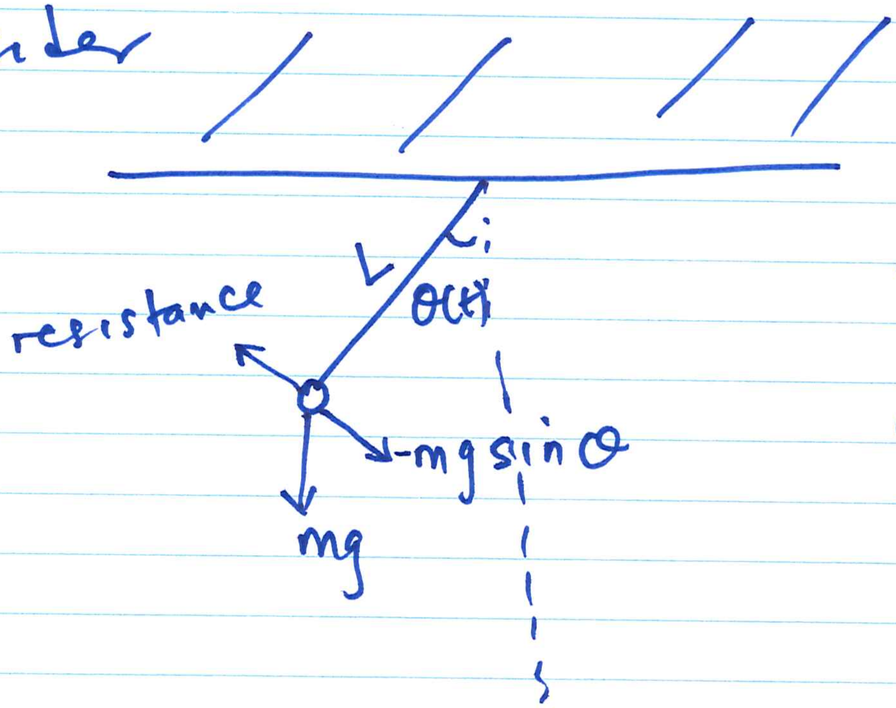
where $\vec{Y}_0$ is an equilibrium solution.

(iii) Determine the eigenvalues of the Jacobian matrix $D\vec{F}(\vec{Y}_0)$ and use it to understand the behaviour of the system close to $\vec{Y}_0$.

(iv) use the eigenvalues and eigenvectors of $D\vec{F}(\vec{Y}_0)$ to sketch the <u>phase portrait</u> of the solution of the system close to $\vec{Y}_0$.

Example: Damped Nonlinear Pendulum

consider



$$v = L\frac{d\theta}{dt}$$

$$a = L\frac{d^2\theta}{dt^2}$$

damping force $= -\mu m L \dfrac{d\theta}{dt}$, $\mu > 0$ constant

By Newton's second law

ma = sum of forces

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta - \mu m L\frac{d\theta}{dt}$$

divide through by mL

$$\theta'' + \frac{g}{L}\sin\theta + \mu\theta' = 0$$

$$\theta'' + \mu\theta' + \frac{g}{L}\sin\theta = 0$$

Let us convert the ODE to system of 1st order ODE.

Let $y_1 = \theta$, $y_2 = \theta'$, then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ -\frac{g}{L}\sin(y_1) - \mu y_2 \end{pmatrix}$$

Let
$$f(y_1, y_2) = y_2$$
$$h(y_1, y_2) = -\frac{g}{L}\sin(y_1) - \mu y_2$$

To find the equilibria, set

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad y_2 = 0$$

$$-\frac{g}{L}\sin(y_1) - \mu y_2 = 0$$

$$\Rightarrow \quad \sin(y_1) = 0$$

$\Rightarrow$ we have

$$y_2 = 0$$

$$\sin(y_1) = 0$$

$\sin(y_1) = 0$

$\Rightarrow y_1 = 0, \pm\pi, \pm 2\pi, \ldots$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\pi \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 2\pi \\ 0 \end{pmatrix}, \ldots$$

down-state | down ---

~~upsta~~

up-state

We shall consider only

$\vec{Y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\vec{Y}_2 = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ (up-state)

down-state

Let us construct our Jacobian matrix.

$$D\vec{F}(\vec{Y}) = \begin{pmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos(y_1) & -\mu \end{pmatrix}$$

Near $\vec{Y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$D\vec{F}(\vec{Y}_1) = \begin{pmatrix} 0 & 1 \\ -g/L & -\mu \end{pmatrix}$$

For the eigenvalues

$$\lambda^2 + \mu\lambda + g/L = 0$$

$$\lambda = \frac{-\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4g/L}$$

case I : $\mu^2 > 4g/L$ (high friction)
(over damped)

$\Rightarrow$ λ is real and negative

$\Rightarrow$ $\vec{Y}_1$ is a $\boxed{\text{stable node.}}$

case II : $\mu^2 < 4g/L$ (low friction)
(under-damped)

$\Rightarrow$ λ is complex

Let $\lambda = \alpha \pm i\beta$

$\Rightarrow$ $\alpha = -\mu/2$ $\therefore$ $\vec{Y}_1$ is $\boxed{\text{Stable spiral}}$.

Case III : $\mu^2 = 4g/L$ (critically damped)

$\Rightarrow \lambda = -\mu/2$ (~~req~~ repeated eigenvalues / ~~defficient~~ defective matrix)

$\vec{Y}_1$ is ~~spir~~ $\boxed{\text{improper stable node}}$

Near $\vec{Y}_2 = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$

$$DF(\vec{Y}_2) = \begin{pmatrix} 0 & 1 \\ g/L & -\mu \end{pmatrix}$$

$$\lambda^2 + \mu\lambda - g/L = 0$$

$$\lambda = -\mu/2 \pm \frac{1}{2}\sqrt{\mu^2 + 4g/L}$$

$\Rightarrow$ ~~Real distinct~~

$\lambda_1 > 0$ and $\lambda_2 < 0$

$\Rightarrow \vec{Y}_2$ is a $\boxed{\text{saddle point}}$. $\boxed{\text{unstable}}$

Return to $\vec{Y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Suppose we have

$$\lambda = -\mu/2 \pm \frac{1}{2}\sqrt{\mu^2 - 4g/L}$$

Suppose there is no damping, i.e $\mu = 0$

$$\Rightarrow \lambda = \pm \frac{1}{2}\sqrt{-4g/L} = \pm \frac{1}{2} i \sqrt{4g/L}$$

$\Rightarrow$ Continuous oscillations.

$\therefore$ We have that the linear system has a <u>center point</u> at $\vec{Y}_1$.

Since we ~~have~~ linearized a nonlinear system which involves ignoring some nonlinear terms, we cannot say for sure that the nonlinear system ~~has~~ also has a periodic solution ~~at~~ close to $\vec{Y}_0$.