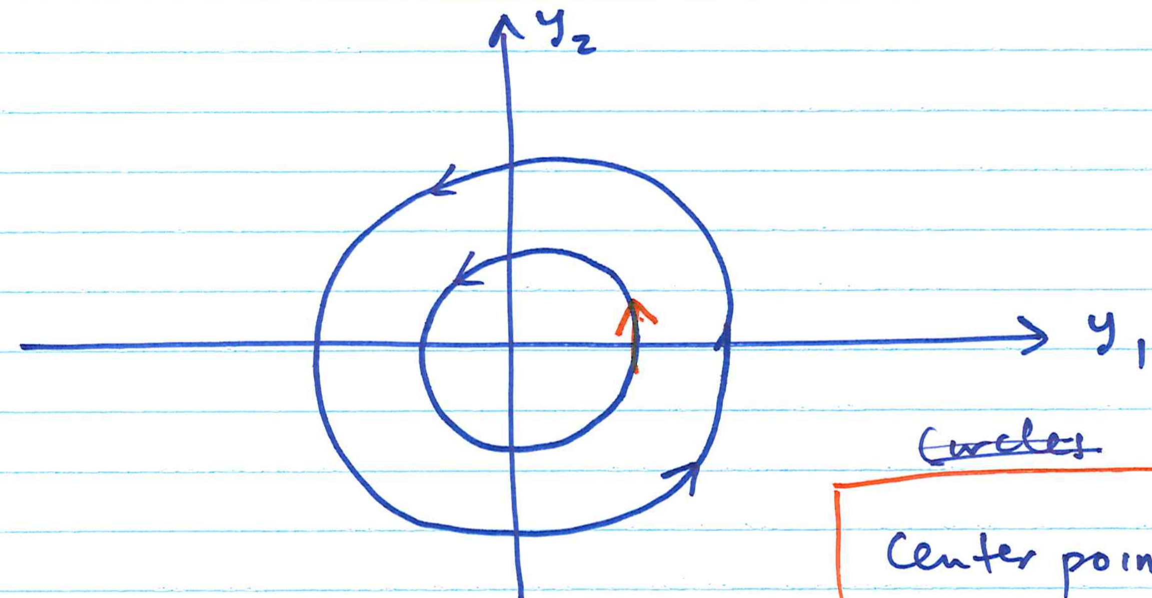
(b) ~~Case~~ $\alpha = 0$. $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$

Example: $\vec{y}\,'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{y}(t)$.

$\lambda_1 = i$ and $\lambda_2 = -i$



~~Circles~~

$\boxed{\text{Center point} \;\Big|\; \text{ellipses} \quad \text{Neutrally stable}}$

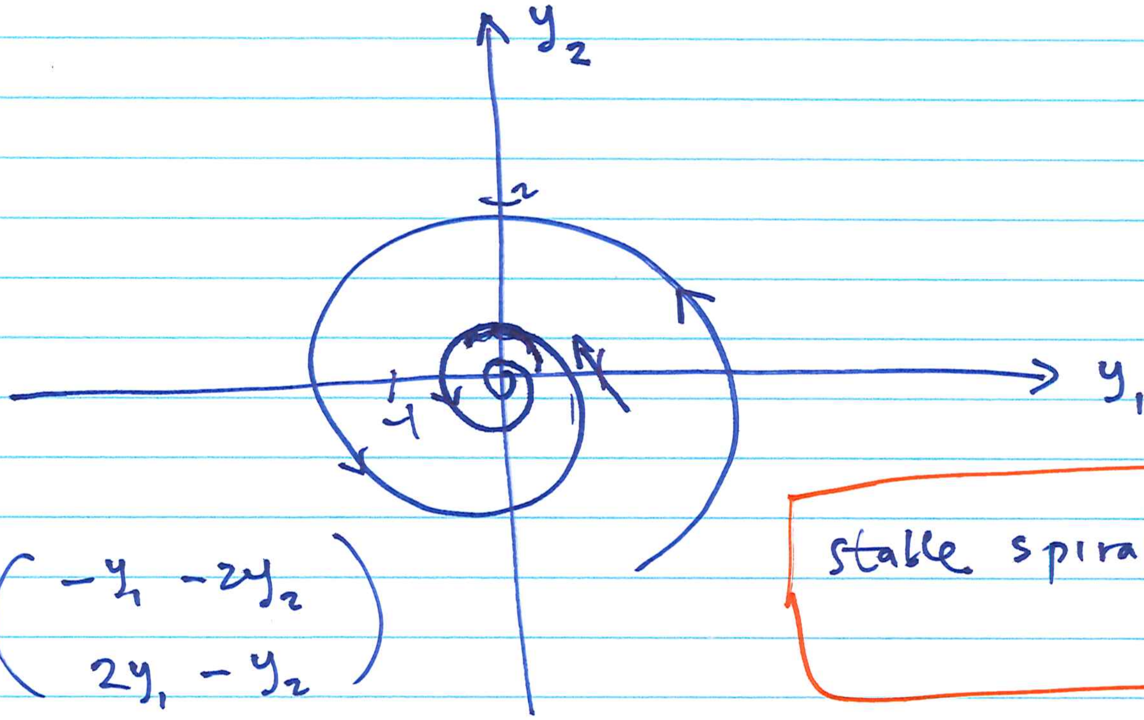
$\vec{F} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$

$(1,0)$, $\vec{F} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

ⓒ $\alpha < 0$,

Example: $\vec{y}\,'(t) = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = -1 + i2$ and $\lambda_2 = -1 - i2$



$$\vec{F} = \begin{pmatrix} -y_1 - 2y_2 \\ 2y_1 - y_2 \end{pmatrix}$$

stable spiral / spiral sink

$(1, 0)$

$$\vec{F} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

FUNDAMENTAL MATRIX

Example: Solve

$$\vec{y}'(t) = A\vec{y}(t)$$

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

For the eigenvalues, $\quad \lambda^2 - tr(A)\,\lambda + det(A) = 0$

$$\lambda^2 - tr(A)\,\lambda + det(A) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_1 = -1, \quad \text{and} \quad \lambda_2 = 3.$$

For $\lambda_1 = -1$,

$$(A - \lambda I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 3$, $\quad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore$ The general solution is

$$\vec{y}(t) = c_1 e^{-t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\2\end{pmatrix}$$

Let $\vec{y}^{(1)}(t) = e^{-t}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix}$

and $\vec{y}^{(2)}(t) = e^{3t}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}$

Let us use this solutions to construct a matrix

$$\Psi = \begin{pmatrix} | & | \\ \vec{y}^{(1)} & \vec{y}^{(2)} \\ | & | \end{pmatrix}$$

For our example:

$$\Psi = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$$

Linear independence

The solutions $\vec{y}^{(1)}(t)$ and $\vec{y}^{(2)}(t)$ of a Linear system are linearly independent for a value of t if

$$\det\left(\Psi(t)\right) \neq 0$$

This determinat is called the Wronskian of the 2 solutions and it is denoted by

$$W\left[\vec{y}^{(1)}, \vec{y}^{(2)}\right] = \det\left(\Psi(t)\right)$$

Let us compute the wronskian for our example.

$$W\left(\vec{y}^{(1)}, \vec{y}^{(2)}\right) = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix}$$

$$= e^{-t}\left(2e^{3t}\right) - \left(-2e^{-t}\right)e^{3t}$$

$$= 2e^{2t} + 2e^{2t} = 4e^{2t} \neq 0$$

for all t.

$\Rightarrow W\left[\vec{y}^{(1)}, \vec{y}^{(2)}\right] \neq 0 \quad \forall t$.

$\Rightarrow$ The solutions $\vec{y}^{(1)}(t)$ and $\vec{y}^{(2)}(t)$ are linearly independent for all values of t.

Any set of linearly independent solutions of the system $\vec{y}\,'(t) = A\vec{y}(t)$ forms a fundamental set of solutions of the system.

Since $\vec{y}^{(1)}(t)$ and $\vec{y}^{(2)}(t)$ are linearly independent, they form a fundamental set of solutions for the system.

And the matrix

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$$

is called ~~the~~ a fundamental matrix of the system.