Last class

We looked at nonlinear pendulum and stopped at the case ~~we~~ where there is no friction.

$$\Rightarrow \quad \mu = 0$$

$$\therefore \quad \lambda = \pm i\sqrt{g/L}$$

The linear system has periodic solution. But we cannot conclude that the nonlinear system also has periodic solution close to $\vec{Y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now, let us show that the nonlinear system has continuous oscillation close to $\vec{Y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

From Newton's 2nd law,

$$\theta'' + \mu\theta' + g/L \sin\theta = 0$$

If $\mu = 0$,

$$\Rightarrow \quad \theta'' + g/L \sin\theta = 0$$

multiply by θ',

$$\theta'\theta'' + \frac{g}{L}\sin\theta\,\theta' = 0$$

integrate with respect to t,

$$\frac{(\theta')^2}{2} + \left(-\frac{g}{L}\right)\cos\theta = c_1$$

multiply ~~through~~ through by mL^2

$$\frac{1}{2}m(L\theta')^2 + (-gmL)\cos\theta = \underbrace{mL^2 c_1}_{c_2}$$

$$\underbrace{\frac{1}{2}m(L\theta')^2}_{\text{"kinetic energy"}} + \underbrace{(-gmL)\cos\theta}_{\text{"potential energy"}} = c_2$$

$\Rightarrow$ Total energy is constant

$\therefore$ We have continuous ~~cons~~ oscillations.

Example: Epidemic models

Acute infectious diseases

- epidemic last for a few months
- Eg flu
- no need to include demographics (birth, death, immigration and emigration)

Chronic infectious diseases

- epidemic last for years.
- Eg: HIV, TB, Hepatitis C
- it's important to include demographics.

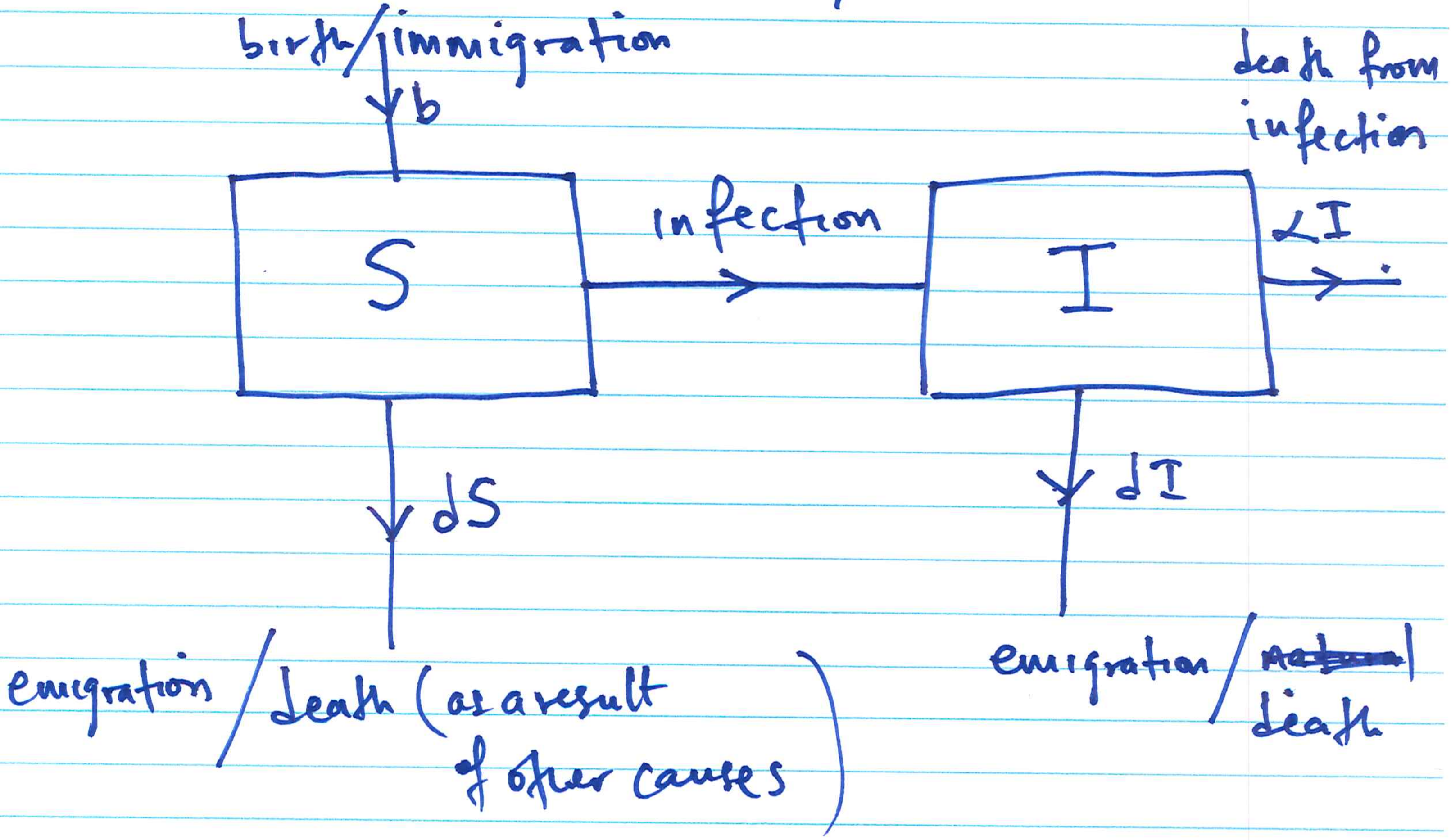
An Example of a model for HIV epidemic.

Consider a population (within a city) ~~suscept~~ at high risk for HIV infection.

Divide this population into two groups:

Susceptible individuals, $S(t)$

Infected individuals, $I(t)$



Define

$\beta = \left(\text{rate of risky } \cancel{\text{constant}} \text{ contact}\right) \times \left(\text{prob. of transmission per contact}\right)$

New infection: $\dfrac{\beta S I}{I+S}$

$d > 0$ constant

$$\left.\begin{aligned} \frac{dS}{dt} &= b - \frac{\beta S I}{I+S} - dS \\ \frac{dI}{dt} &= \frac{\beta S I}{I+S} - \alpha I - dI \end{aligned}\right\} \text{—} (*)$$

$$\left.\begin{aligned} \frac{dS}{dt} &= -\frac{\beta S I}{I+S} + b - dS \\ \frac{dI}{dt} &= \frac{\beta S I}{I+S} - (\alpha + d) I \end{aligned}\right\} \text{—} (*1)$$

$d > 0$, $\alpha > 0$ are rates.

Linear analysis

First, find the steady-state solutions.

Set $ds/dt = 0$, $dI/dt = 0$

$$\Rightarrow \quad -\frac{\beta S I}{S+I} + b - dS = 0 \qquad \text{——— (2)}$$

$$I\left[\frac{\beta S}{S+I} - (\alpha+d)\right] = 0 \qquad \text{——— (3)}$$

From (3), $I_1 = 0$

$$\frac{\beta S}{S+I} - (\alpha+d) = 0$$

$$\Rightarrow \quad I_2 = S\left[\frac{\beta-(\alpha+d)}{(\alpha+d)}\right]$$

put $I_1 = 0$ into (2), we get

$$S_1 = b/d$$

put I_2 into (2) and this will give $S_2 = \frac{b}{\beta - \alpha}$

$\therefore$ The equilibria are

(i) $(S_1, I_1) = \left(b/d, 0\right)$ (disease free equilibrium)

(ii) $(S_2, I_2) = \left(\frac{b}{\beta - \alpha}, \frac{b(\beta - (\alpha + d))}{(\alpha + d)(\beta - \alpha)}\right)$

(endemic equilibrium)

where $\beta - \alpha \neq 0$

$\beta > \alpha$ and $\beta > \alpha + d$

Next, construct the Jacobian matrix.

Let $f(S,I) = -\dfrac{\beta I S}{S+I} + b - dS$

$g(S,I) = \dfrac{\beta I S}{S+I} - (\alpha+d)I$

$$D\vec{F} = \begin{pmatrix} \dfrac{\partial f}{\partial S} & \dfrac{\partial f}{\partial I} \\[2ex] \dfrac{\partial g}{\partial S} & \dfrac{\partial g}{\partial I} \end{pmatrix}$$

$$D\vec{F} = \begin{pmatrix} \dfrac{-\beta I^2}{(S+I)^2} - d & \dfrac{-\beta S^2}{(S+I)^2} \\[3ex] \dfrac{\beta I^2}{(S+I)^2} & \dfrac{\beta S^2}{(S+I)^2} - (\alpha+d) \end{pmatrix}$$

For disease free equilibrium,

$$(S_1, I_1) = \left(b/d, 0\right)$$

$$D\vec{F}\left((S_1, I_1)\right) = \begin{pmatrix} -d & -\beta \\ 0 & \beta - (\alpha + d) \end{pmatrix}$$

The eigenvalues are

$$\lambda_1 = -d \quad \text{and} \quad \lambda_2 = \beta - (\alpha + d)$$

$$\lambda_1 = -d < 0 \quad (\text{since } d > 0).$$

For this equilibrium to be stable, we need $\lambda_2 < 0$

$$\Rightarrow \quad \beta - (\alpha + d) < 0$$

$$\boxed{\frac{\beta}{\alpha + d}} < 1$$

$\longleftarrow R_0$ (basic reproductive number)

$\therefore$ For the ~~dea~~ disease to die out ~~of~~ the population, we need $R_0 < 1$.