Example: Solve

$$y'' + y = \cos(t)$$

$$y(0) = 0, \quad y'(0) = 0$$

First, take L-T of the ODE.

$$L[y'' + y] = L[\cos(t)]$$

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + Y(s) = \frac{s}{s^2+1}$$

applying our initial conditions

$$y(0) = 0, \quad y'(0) = 0$$

$$Y(s)\left(s^2+1\right) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s^2+1)} \cdot \frac{1}{(s^2+1)}$$

$$Y(s) = L[\underbrace{\cos(t)}_{f}] \cdot L[\underbrace{\sin(t)}_{g}]$$

By ~~convolution of~~ L-T of convolution, we have

$$y(t) = \cos(t) * \sin(t)$$

By definition of convolution,

$$y(t) = \cos(t) * \sin(t) = \int_0^t \cos(t-x)\sin(x)\,dx$$

$$= \int_0^t \left[\cos(t)\cos(x) + \sin(t)\sin(x)\right]\sin(x)\,dx$$

$$= \cos(t)\int_0^t \cos(x)\sin(x)\,dx + \sin(t)\int_0^t \sin^2(x)\,dx \quad \text{—} \; (1)$$

Recall

$$\cos(x)\sin(x) = \frac{1}{2}\sin(2x) \quad \text{———} \; (2)$$

~~$\sin^2(x) =$~~

Recall,

$$\cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin^2(x) = \cancel{\cos(2x)} \; \cos^2(x) - \cos(2x) \quad \text{—} \; (*1)$$

Also

$$\sin^2(x) + \cos^2(x) = 1$$

$$\cos^2(x) = 1 - \sin^2(x)$$

$$\sin^2(x) = 1 - \sin^2(x) - \cos(2x)$$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right) \quad \text{———} \; (3)$$

putting (2) and (3) into (1), we have

$$y(t) = \cos(t)\int_0^t \frac{1}{2}\sin(2x)\,dx + \sin(t)\int_0^t \frac{1-\cos(2x)}{2}\,dx$$

$$= \frac{\cos(t)}{2}\left[-\frac{\cos(2x)}{2}\right]_0^t + \frac{\sin(t)}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^t$$

$$y(t) = -\frac{\cos(t)\cos(2t)}{4} + \frac{1}{4}\cos(t) + \frac{t}{2}\sin(t) - \frac{\sin(t)\sin(2t)}{4}$$

$$= -\frac{1}{4}\underbrace{\left[\cos(t)\cos(2t) + \sin(t)\sin(2t)\right]}_{\cos(t)} + \frac{1}{4}\cos(t) + \frac{t}{2}\sin(t)$$

$$= \cancel{-\frac{1}{4}\cos(t)} \cancel{+\frac{1}{4}\cos(t)} + \frac{t}{2}\sin(t)$$

$$y(t) = \frac{1}{2}t\sin(t)$$

Consider the forced system,

$$ay'' + by' + cy = g(t)$$

$$y(0) = 0, \quad y'(0) = 0$$

Taking the L.T of the problem,

$$as^2 Y(s) + bs Y(s) + cY(s) = L[g(t)]$$

$$Y(s) = \frac{1}{(as^2 + bs + c)} \cdot L[g(t)]$$

Use convolution to take the inverse L.T.

$$y(t) = L^{-1}\left[\frac{1}{(as^2 + bs + c)}\right] * g(t)$$

Suppose $g(t) = \delta(t)$ (impulse function)

$$Y(s) = \frac{1}{(as^2 + bs + c)} \cdot L[\delta(t)] = \frac{1}{as^2 + bs + c}$$

Since,

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} \, dt = 1$$

$\delta(t - t_0)$, Here $t_0 = 0$

The function $\dfrac{1}{as^2+bs+c}$ is called the

<u>transfer function</u> of the system while

$L^{-1}\left[\dfrac{1}{as^2+bs+c}\right]$ is called the

<u>impulse response function</u> of the system.

This function is the response of the system to the impulse function (or to the forcing function).

<u>Example:</u> Solve

$$y'' + 9y = \delta(t)$$
$$y(0) = 0, \quad y'(0) = 0$$

The L.T of the ODE,

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + 9Y(s) = L[\delta(t)]$$

applying the I.Cs., we have

$$Y(s)\,(s^2+9) = 1.$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2+9}$$

$\Rightarrow$ the transfer function of the system is $\frac{1}{s^2+9}$ while the impulse response function is

$$L^{-1}\left[\frac{1}{s^2+9}\right].$$

$$y(t) = L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin(3t)$$

Observation: solve

$$y'' + 9y = 0$$

$$y(0)=0, \quad y'(0)=0$$

$$\Rightarrow \quad m^2+9=0$$

$$\Rightarrow \quad m = \pm i3$$

$$y(t) = c_1\cos(3t) + c_2\sin(3t)$$

applying the initial conditions, we have

$$c_1 = 0, \quad c_2 = 0.$$

$$\Rightarrow \quad y(t) = 0$$

This ~~implies~~ implies that the only way we can have a nontrivial solution is ~~if~~ if there is an input to the system (that is, a forcing function).

<u>Example</u>: Solve the IVP

$$y' + y - \int_0^t y(x) \sin(t-x)\, dx = -\sin t$$

$$y(0) = 1$$

$$L[y'] + L[y] - L\left[\int_0^t y(x) \sin(t-x)\, dx\right] = -L[\sin t]$$

$$\left(sL[y] - y(0)\right) + L[y] - L\left[y(t) * \sin(t)\right] = -\frac{1}{s^2+1}$$

$$sL[y] - 1 + L[y] - L[y] \cdot L[\sin(t)] = -\frac{1}{s^2+1}$$

$$L[y](s+1) - 1 - \frac{L[y]}{s^2+1} = -\frac{1}{s^2+1}$$

$$L[y]\left[(s+1) - \frac{1}{s^2+1}\right] = 1 - \frac{1}{s^2+1}$$

$$\mathcal{L}[y]\left[\frac{\cancel{s^2}+s(s^2+1)+s^2\cancel{+1}\cancel{-1}}{s^2+1}\right]=1-\frac{1}{s^2+1}$$

$$\mathcal{L}[y]\left(\frac{s(s^2+1)+s^2}{\cancel{(s^2+1)}}\right)=\frac{s^2+1-1}{\cancel{(s^2+1)}}$$

$$\mathcal{L}[y]=\frac{s^2}{s(s^2+1)+s^2}=\frac{s}{s^2+1+s}$$

$$=\frac{s}{s^2+s+1}=\frac{s}{\left(s^2+s+\frac{1}{4}\right)-\frac{1}{4}+1}$$

$$=\frac{s}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{s}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}[y]=\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{\sqrt{3}}\frac{\frac{1}{2}\cdot\sqrt{3}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$y(t)=e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$