Last class

We looked at the system

$$\vec{Y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{Y}(t)$$

whose solution is

$$\vec{Y}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let

$$\vec{Y}^{(1)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$Y^{(2)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$

Then

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$$

We showed that $\vec{Y}^{(1)}(t)$ and $\vec{Y}^{(2)}(t)$ are independent, i.e

$$W(\vec{Y}^{(1)}, \vec{Y}^{(2)}) = \det(\Psi) \neq 0 \ \forall t$$

Remark

The Wronskian is either zero $\forall\, t$ (i.e. when $\vec{y}^{(1)}$ and $\vec{y}^{(2)}$ are linear dependent) or not zero at all $\forall\, t$ (when they are independent).

Return to the general solution,

$$\vec{y}(t) = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}}_{\Psi} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\Rightarrow$

$$\vec{y}(t) = \Psi(t)\,\vec{c} \qquad (*)$$

where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Suppose we are given

$$\vec{Y}(0) = \vec{Y}_0$$

~~pf.~~ apply the I.C. to (*) to find $\vec{c}$.

$$\vec{Y}_0 = \Psi(0)\,\vec{c}$$

$$\vec{c} = \left(\Psi(0)\right)^{-1} \vec{Y}_0$$

put $\vec{c}$ into (*),

$$\Rightarrow \boxed{\vec{Y}(t) = \Psi(t)\left(\Psi(0)\right)^{-1} \vec{Y}_0} \text{ — (box 1)}$$

For example, given $\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Psi(0) = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$$

$$\left(\Psi(0)\right)^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{pmatrix}$$

$$(\Psi(0))^{-1}\vec{Y}_0 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{Y}(t) = \Psi(t)\,(\Psi(0))^{-1}\,\vec{Y}_0$$

$$= \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} e^{-t} + 3e^{3t} \\ -2e^{-t} + 6e^{3t} \end{pmatrix}$$

$$\vec{Y}(t) = \frac{1}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \frac{1}{4}\begin{pmatrix} 3 \\ 6 \end{pmatrix} e^{3t}.$$

Matrix exponential

Recall, if we have a scalar ODE

$$y'(t) = a\,y(t),$$

the solution is

$$y(t) = C\,e^{at}$$

Now,

consider the vector equation

$$\vec{y}\,'(t) = A\,\vec{y}(t)$$

Can we write the solution as

$$\vec{y}(t) = \vec{C}\,e^{At} \quad ?$$

YES!! 

consider the Taylor expansion,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Matrix exponential

Let A be an $n \times n$ matrix (real or complex),

$$e^{tA} = I + (tA) + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots$$

Que: what is $\frac{d}{dt} e^{tA}$?

$$\Rightarrow \frac{d e^{tA}}{dt} = A e^{tA}$$

You can derived this by doing ~~using~~ Taylor expansion of e^{tA} and then differentiating w.r.t t.

Let us compute e^{tA},

(1) Suppose A is a diagonal matrix,

E.g let $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$

$$e^{tA} = I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{pmatrix} + \frac{t^3}{3!} \begin{pmatrix} a_1^3 & 0 \\ 0 & a_2^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + ta_1 + \frac{t^2 a_1^2}{2!} + \frac{t^3 a_1^3}{3!} + \cdots & 0 \\ 0 & 1 + ta_2 + \frac{t^2}{2!} a_2^2 + \frac{t^3}{3!} a_2^3 + \cdots \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{pmatrix}$$

(1) Suppose A is diagonalizable (i.e A is not a diagonal matrix but diagonalizable)

$\Rightarrow$ A can be written as the product of 3 matrices, i.e

$$A = PDP^{-1}$$

where P is a matrix whose columns are eigenvectors of A

and D is a diagonal matrix whose entries are the eigenvalues of A (in the same order as in P).

Now, let us compute e^{tA}

$$e^{tA} = e^{t(PDP^{-1})}$$

$$= I + t(PDP^{-1}) + \frac{t^2}{2!}(PDP^{-1})^2 + \frac{t^3}{3!}(PDP^{-1})^3 + \cdots$$

But

$$(PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1}$$

$$= PD^2P^{-1}$$

$$e^{tA} = I + t(PDP^{-1}) + \frac{t^2}{2!}(PD^2P^{-1}) + \frac{t^3}{3!}(PD^3P^{-1}) + \cdots$$

$$= P\left[I + tD + \frac{t^2D^2}{2!} + \frac{t^3}{3!}D^3 + \cdots\right]P^{-1}$$

$$= P e^{tD} P^{-1}$$

$$\therefore \quad e^{tA} = P e^{tD} P^{-1}$$

Return to our example.

We have

$$\lambda_1 = -1, \qquad \lambda_2 = 3$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}, \quad P^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$e^{tA} = P e^{tD} P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$e^{tA} = \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & -e^{-t} + e^{3t} \\ -4e^{-t} + 4e^{3t} & 2e^{-t} + 2e^{3t} \end{pmatrix}$$

$\therefore$ The general solution is

$$\vec{Y}(t) = e^{tA} \vec{C}$$

Let us find $\vec{C}$. At $t=0$,

$$\vec{Y}(0) = \vec{C} = \vec{Y_0}$$

Observe :

At $t=0$

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This is why $\vec{Y}(0) = I\vec{c} = \vec{Y}_0 \Rightarrow \vec{c} = \vec{Y}_0$

$$\boxed{\vec{Y}(t) = e^{tA}\vec{Y}_0} \quad \text{— box 2.}$$

Comparing box 1 and box 2, we observe that

$$\boxed{e^{tA} \equiv \Psi(t)\left(\Psi(0)\right)^{-1}}$$

** New **

continuing . . .

$$\vec{Y}(t) = e^{tA}\vec{Y}_0 \quad , \quad \vec{Y}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2e^{-t}+2e^{3t} & -e^{-t}+e^{3t} \\ -4e^{-t}+4e^{3t} & 2e^{-t}+2e^{3t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2e^{-t}+2e^{3t}-e^{-t}+e^{3t} \\ -4e^{-t}+4e^{3t}+2e^{-t}+2e^{3t} \end{pmatrix}$$

$$\vec{Y}(t) = \frac{1}{4}\begin{pmatrix} e^{-t}+3e^{3t} \\ -2e^{-t}+6e^{3t} \end{pmatrix}$$

$$\vec{Y}(t) = \frac{1}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + \frac{1}{4}\begin{pmatrix} 3 \\ 6 \end{pmatrix}e^{3t}$$

This is exactly the same as what we got using the other method (fundamental matrix).