Example: Solve

$$(3xy + y^2) + (x^2 + xy)\, dy/dx = 0 \quad \text{--- } (1)$$

$M(x,y) = 3xy + y^2$, $M_y = 3x + 2y$

$N(x,y) = x^2 + xy$, $N_x = 2x + y$

Let check if $\exists$ an I.F that is a function of x only

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy}$$

$$= \frac{x + \cancel{y}}{x \cancel{(x+y)}} = \frac{1}{x}$$

To get the I.F, we solve the ODE

$$\frac{dh}{dx} = \frac{1}{x} \cdot h$$

$$\int \frac{dh}{h} = \int \frac{1}{x}\, dx + C_1$$

$$\ln(h) = \ln(x) + C_1$$

$$h(x) = C_2 e^{\ln(x)} = C_2 x$$

$\boxed{\text{Set } C_2 = 1}$

$$h(x) = x$$

multiply through ① by h.

$$(3x^2y + xy^2) + (x^3 + x^2y)\,dy/dx = 0$$

$M = 3x^2y + xy^2$

$N = x^3 + x^2y \qquad N_x = 3x^2 + 2xy$

<u>check if equation is exact.</u>

$$M_y = 3x^2 + 2xy \overset{?}{=} N_x \quad \checkmark$$

equation is exact.

We want find a function $\psi(x,y)$

such that

$$\frac{\partial \psi}{\partial x} = M \quad , \quad \frac{\partial \psi}{\partial y} = N$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = 3x^2 y + xy^2$$

$$\psi(x,y) = x^3 y + \frac{x^2}{2} y^2 + \gamma_1(y) \text{ ——— } (2)$$

$$\frac{\partial \psi}{\partial y} = x^3 + x^2 y$$

$$\Rightarrow \psi(x,y) = x^3 y + x^2 \frac{y^2}{2} + \gamma_2(x) \text{ ——— } (3)$$

Comparing (2) and (3), we can set

$$\gamma_1(y) = \gamma_2(x) = 0.$$

$$\psi(x,y) = x^3 y + x^2 y^2/2$$

∴ our solution is

$$x^3 y + x^2 \frac{y^2}{2} = C.$$

Numerical Approximation of Solutions of ODEs.

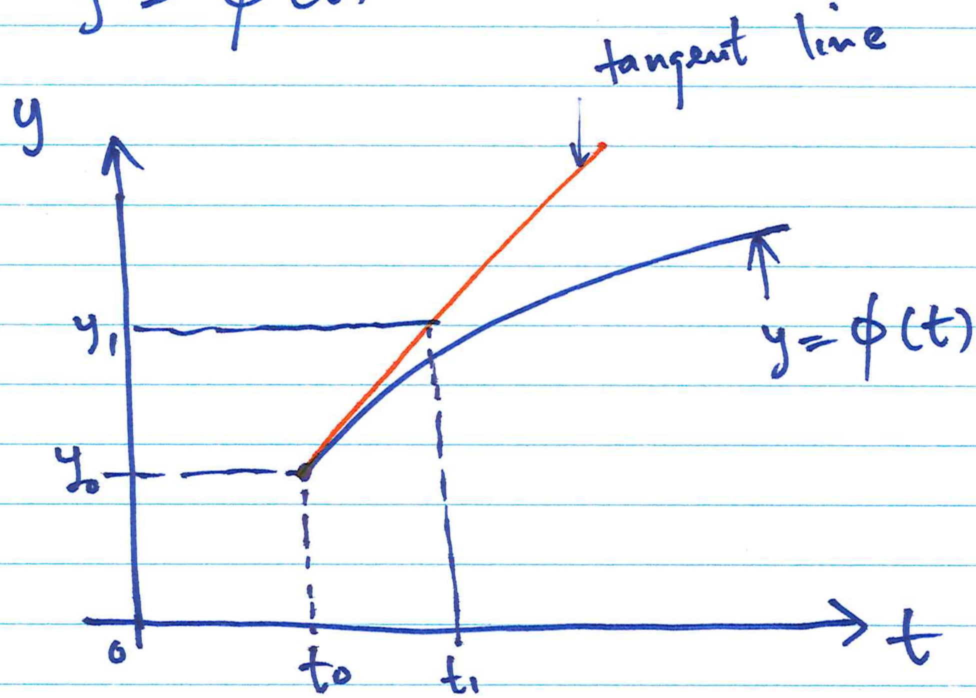
Euler's method (Tangent line method)

Given an IVP

$$y' = f(t, y), \quad y(t_0) = y_0 \quad \text{—— (1)}$$

Suppose the analytic solution is

$$y = \phi(t)$$



$$y' = \left.\frac{d\phi}{dt}\right|_{t=t_0}$$

The equation of the tangent line is

$$(y - y_0) = \left.\frac{dy}{dt}\right|_{t_0} (t - t_0)$$

$$\left.\frac{dy}{dt}\right|_{t_0} = f(t_0, y_0)$$

$$(y - y_0) = f(t_0, y_0)(t - t_0)$$

$$y = y_0 + f(t_0, y_0)(t - t_0).$$

Goal: use ~~the~~ tanglent lines to find an approximate solution to the IVP.

To get y_1 at point t_1

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

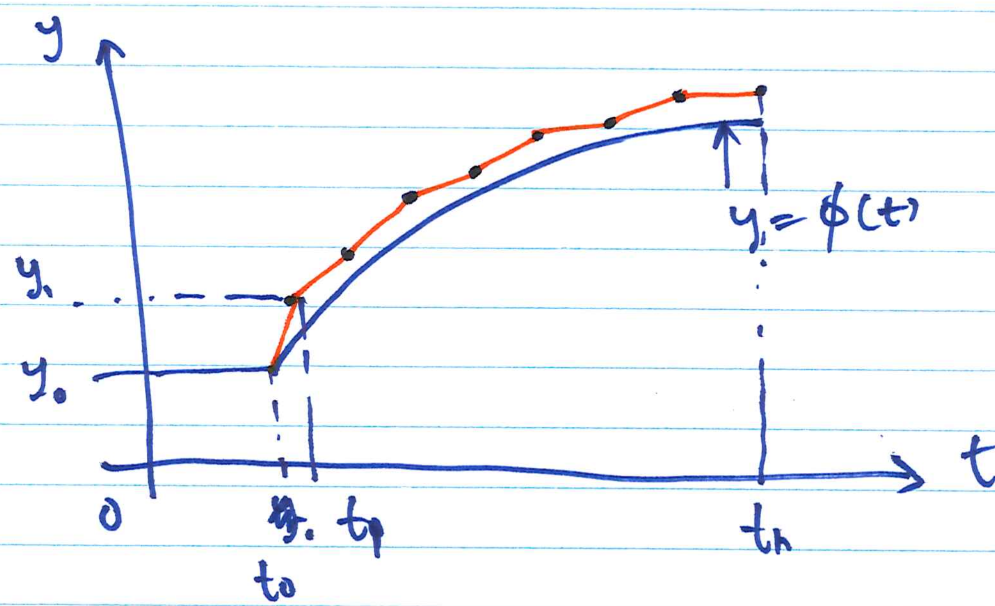
and for y_2 at t_2,

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

continuing this way, we have

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$

for the $(n+1)^{th}$ approximation.



Assuming the step size in time is uniform

say $t_{n+1} - t_n = h$

$t_{n+1} = h + t_n$

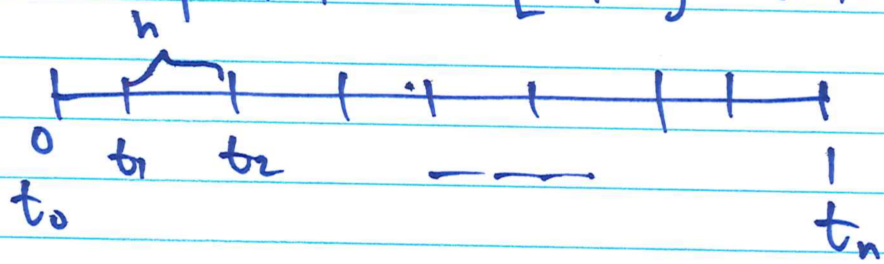
Then the Euler's method is given by

$$\boxed{y_{n+1} = y_n + h f(t_n, y_n) \qquad n = 0, 1, 2, \ldots}$$

Example: Solve $y' = (2-t)y$, $y(0) = 1$

for $0 \le t \le 1$

First, partition $[0, 1]$ into equal sub intervals



take $h = 0.1$

$t_0 = 0$, $y_0 = 1$, $f(t, y) = (2-t)y$

Euler's method

$$y_{n+1} = y_n + h f(t_n, y_n)$$

when $n = 0$, we have

$$y_1 = y_0 + h f(t_0, y_0)$$

$$y_1 = 1 + 0.1\left[(2-0)1\right] = 1.2$$

when $n = 1$,

$$y_2 = y_1 + h f(t_1, y_1)$$

$$= 1.2 + 0.1\left[(2-0.1)1.2\right] = 1.428$$

$$y_3 = y_2 + h\,f(t_2, y_2)$$
$$= 1.428 + 0.1\left[(2-0.2)\,1.428\right]$$
$$= \underline{\underline{1.6850}}$$