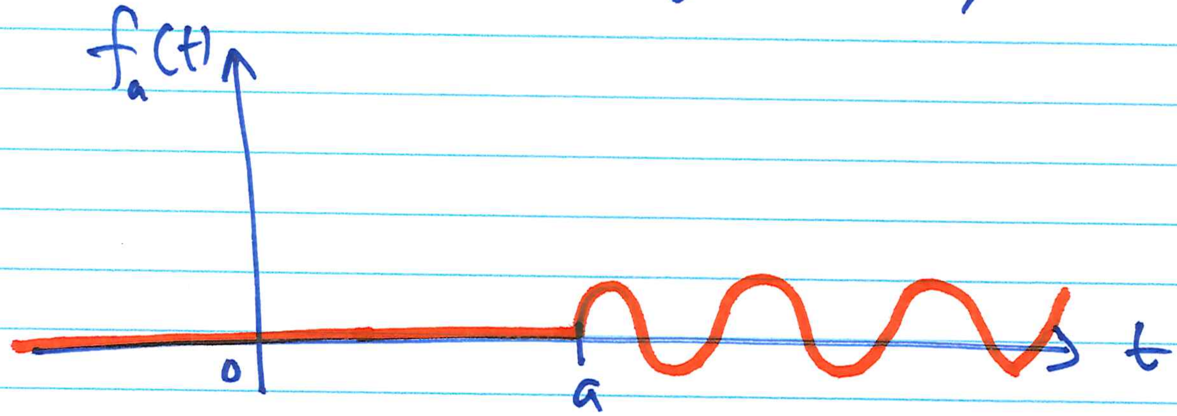
Example of a shifted function

$$f_a(t) = u(t-a)\sin(t-a)$$

$$f_a(t) = \begin{cases} 0, & t < a \\ \sin(t-a), & t \geq a \end{cases}$$



Laplace transform of shifted function

$$L\left[u(t-a)f(t-a)\right] = \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt, \quad s>0$$

$$= \cancel{\int_0^a 0 \cdot f(t-a)e^{-st}\,dt}^{\,0} + \int_a^\infty 1 \cdot f(t-a)e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)e^{-st}\,dt$$

Let $u = t - a \Rightarrow t = u + a$

$du = dt$

when $t = a, \quad u = 0$

$t = \infty, \quad u = \infty$

$\therefore$ we have

$$L[u(t-a)f(t-a)] = \int_0^\infty f(u)\, e^{-s(u+a)}\, du$$

$$= \int_0^\infty f(u)\, e^{-us}\, e^{-sa}\, du$$

$$= e^{-sa} \int_0^\infty f(u)\, e^{-su}\, du$$

$$= e^{-sa} L[f(t)]$$

$$\boxed{L[u(t-a)f(t-a)] = e^{-sa} L[f(t)]},$$

(formula 4 on our table)

Example: Find L.T of $f(t) = u(t-a)\sin(t-a)$, $a>0$

$$L[f(t)] = L[u(t-a)\sin(t-a)]$$

$$= e^{-as} L[\sin(t)]$$

$$L[f(t)] = \frac{e^{-as}}{s^2+1}$$

Example: Find $y(t)$ such that

$$L[y] = \frac{s e^{-s}}{s^2+1}$$

$$= e^{-s} \cdot \frac{s}{s^2+1}$$

$$= e^{-s} L[\cos(t)]$$

~~From~~ using formula (4) on the table,

$$y(t) = \cancel{u(t-1)\sin(t-1)}$$

$$= u(t-1)\cos(t-1)$$

L.T of $e^{at} f(t)$

$$L[e^{at} f(t)] = \int_0^\infty (e^{at} f(t))\, e^{-st}\, dt, \quad s > 0$$

$$= \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$$

$$\boxed{L[e^{at} f(t)] = F(s-a), \quad s > a}$$

(formula 2 on table).

Example: Find L.T of $g(t) = e^{6t} \sin(4t)$

$a = 6$, $f(t) = \sin 4t$

$$L[e^{6t} \sin(4t)] = F(s-6)$$

Now, $f(t) = \sin(4t)$

$$L[f(t)] = \frac{4}{s^2 + 4^2} = F(s)$$

$$F(s-6) = \frac{4}{(s-6)^2 + 4^2}$$

Derivative formula

$$L[t^n f(t)] = \int_0^\infty t^n f(t) e^{-st}\, dt$$

$$\boxed{L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}}$$

(formula 5 on table)

Example: Find L.T of $f(t) = t\sin(t)$

$n=1$

$$L[f(t)] = L[t\sin(t)] =$$

$$= (-1)^1 \frac{dY(s)}{ds}$$

$$f(t) = t\,y(t)$$

$$L[y] = Y(s)$$

$$y = \sin(t), \quad L[y] = \frac{1}{s^2+1}$$

$$\frac{dY(s)}{ds} = \left(0 - \frac{2s}{(s^2+1)^2}\right) = -\frac{2s}{(s^2+1)^2}$$

putting everything together, we have

$$L[f(t)] = -\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$$

Example: Solve using L.T

$$y'' + 6y' + 34y = 30\sin(2t) \quad \text{——— } ①$$

$$y(0) = 0, \quad y'(0) = 0 \quad \text{——— } ②$$

First, we take L.T of ①

$$L[y'' + 6y' + 34y] = L[30\sin(2t)]$$

$$L[y''] + 6L[y'] + 34L[y] = 30L[\sin(2t)] \quad *$$

$$L[y''] = \int_0^\infty y'' e^{-st}\,dt = s^2 Y(s) - s y(0) - y'(0)$$

$$L[y'] = \int_0^\infty y' e^{-st}\,dt = s Y(s) - y(0)$$

$$L[\sin(2t)] = \frac{2}{s^2+4}$$

$$L[y] = Y(s)$$

$\therefore$ (*) becomes

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + 6\left(s Y(s) - y(0)\right)$$

$$+ 34 Y(s) = \frac{60}{s^2+4}$$

applying the Initial conditions,

$$y(0) = 0, \quad y'(0) = 0$$

$$s^2 Y(s) + 6s Y(s) + 34 Y(s) = \frac{60}{s^2+4}$$

$$Y(s)\left[s^2 + 6s + 34\right] = \frac{60}{s^2+4}$$

$$Y(s) = \frac{60}{(s^2+4)(s^2+6s+34)}$$

This is our solution in frequency domain, we need to tranform the solution to time domain.

We need to first use partial fractions,

Let

$$\frac{60}{(s^2+4)(s^2+6s+34)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s+34}$$

$$\frac{60}{(s^2+4)(s^2+6s+34)} = \frac{(As+B)(s^2+6s+34) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+6s+34)}$$

$$= \frac{(A+C)s^3 + (6A+B+D)s^2 + (34A+6B+4C)s + (34B+4D)}{(s^2+4)(s^2+6s+34)}$$

Compare coefficients of powers of s on both sides.

For s^3 : $A + C = 0$

For s^2 : $6A + B + D = 0$

For s : $34A + 6B + 4C = 0$

Constants : $34B + 4D = 60$

$$\begin{pmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 60 \end{pmatrix}$$