Example: Solve

(i) $y'' + 5y' + 6y = 0$

let $y = e^{\lambda t}$, $y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6 e^{\lambda t} = 0$$

$$\Rightarrow \quad \lambda^2 + 5\lambda + 6 = 0$$

$$\lambda^2 + 2\lambda + 3\lambda + 6 = 0$$

$$\lambda = -2 \text{ and } \lambda = -3$$

$$\therefore y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

(ii) $y'' - 2y' + 2y = 0$

$$\Rightarrow \quad \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$y(t) = c_1 e^{t} \cos(t) + c_2 e^{t} \sin(t)$$

Damping

Given $y'' + by' + cy = 0$

with a solution of the form $y = e^{\lambda t}$

where

$$\lambda = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$$

If $b \geq 0$ and $c > 0$, then

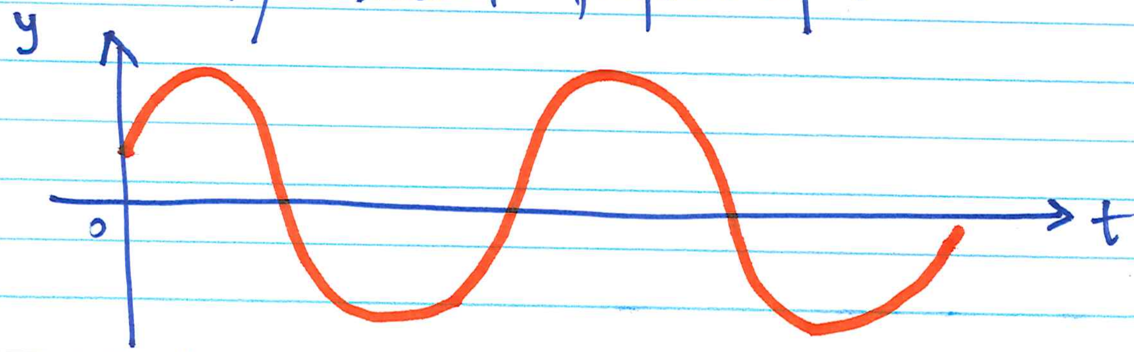
b is called the damping/friction coefficient.

Case I: $b = 0$ (no damping / no friction)

$$\Rightarrow \lambda = \pm \frac{1}{2}\sqrt{-4c} = \pm i\sqrt{c}$$

$$\therefore y(t) = c_1 \cos(\sqrt{c}\, t) + c_2 \sin(\sqrt{c}\, t).$$

~~And~~ ~~the~~ The solution is periodic and $\sqrt{c}$ is its ~~the~~ natural/resonant frequency.



Case II : $b > 0$

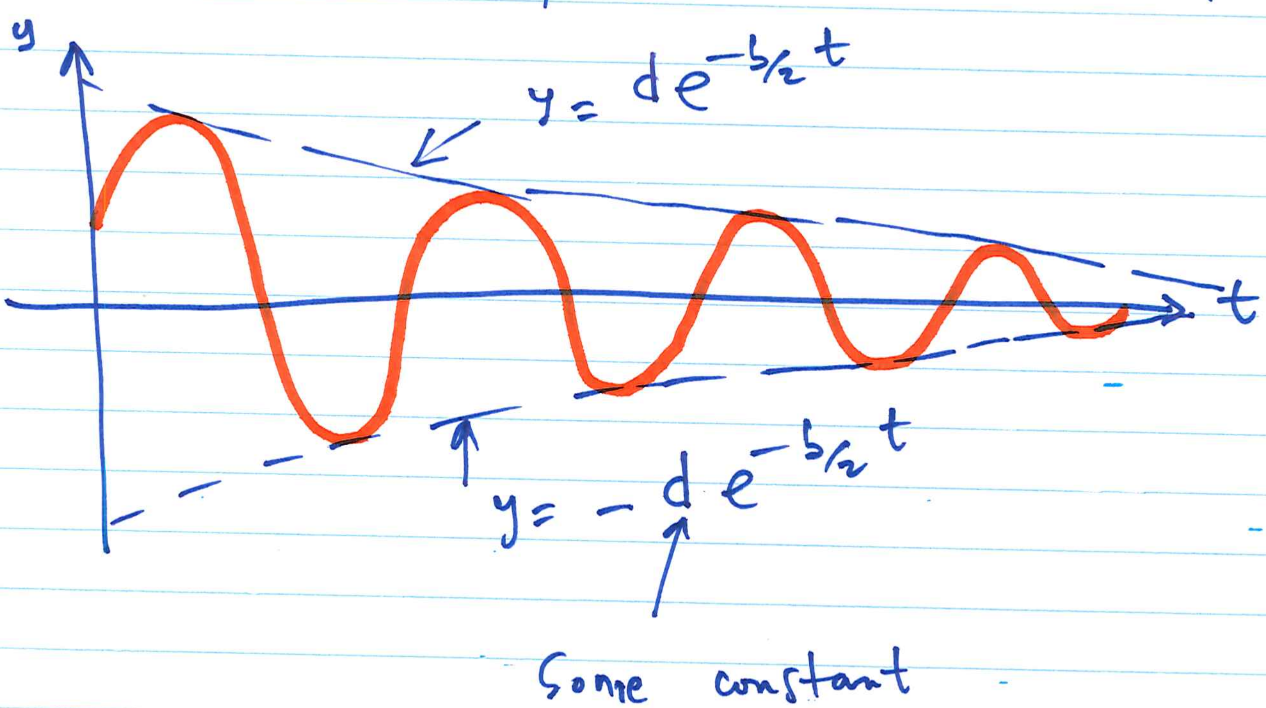
(i) $b^2 < 4c \Rightarrow b^2 - 4c < 0$ (under damping)

$\Rightarrow \lambda$ is complex

Let $\lambda = \alpha \pm i\beta$

$\Rightarrow \alpha = -b/2 \quad , \quad \beta = \frac{1}{2}\sqrt{b^2 - 4c}$

$$y(t) = c_1 e^{-b/2\, t} \cos(\beta t) + c_2 e^{-b/2\, t} \sin(\beta t)$$



(ii) $b^2 = 4c \Rightarrow b^2 - 4c = 0$ (critical damping)

$\Rightarrow \lambda = -b/2$ (repeated roots)

$\therefore y = c_1 e^{-b/2 t} + c_2 t e^{-b/2 t}$

(iii) $b^2 > 4c \Rightarrow b^2 - 4c > 0$ (over damping).

$\Rightarrow \lambda_1$ and λ_2 are real and distinct

$\Rightarrow$ No oscillations.

Return to ~~ea~~ the case where λ is complex.

$b = 0, \quad \lambda = \pm i\beta$.

$\therefore y(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t)$.

~~Again~~ Goal! write the solution in ~~the~~ form

$$y(t) = R \cos(n - m) \quad \text{——— } (*_1)$$

Recall,

$$R\cos(n-m) = R\cos(n)\cos(m) + R\sin(n)\sin(m)$$

compare with

$$y = C_1\cos(\beta t) + C_2\sin(\beta t)$$

$$\cos(n) = \cos(\beta t) \quad , \quad \sin(n) = \sin(\beta t)$$

$$\Rightarrow \quad n = \beta t$$

$$\Rightarrow \quad C_1 = R\cos(m) \quad \text{and} \quad C_2 = R\sin(m)$$

$$C_1^2 + C_2^2 = R^2\cos^2(m) + R^2\sin^2(m)$$

$$= R^2 \underbrace{\left(\cos^2(m) + \sin^2(m)\right)}_{=1}$$

$$\Rightarrow \quad R^2 = C_1^2 + C_2^2$$

$$R = \sqrt{C_1^2 + C_2^2}$$

To get m,

$$\frac{C_2}{C_1} = \frac{R\sin(m)}{R\cos(m)} = \tan(m)$$

$$\Rightarrow \quad m = \arctan\left(\frac{C_2}{C_1}\right)$$

put γ, M, and k into (*1),

$$y(t) = \underbrace{\sqrt{c_1^2 + c_2^2}}_{\text{amplitude}} \cos\left(\underbrace{\beta}_{\substack{\text{angular} \\ \text{frequency}}} t - \arctan\left(\frac{c_2}{c_1}\right)\right)$$