Constant Coefficient Linear systems

Consider the following system of ODE.

$$y_1' = y_1 + 2y_2 \qquad \text{——— (1)}$$
$$y_2' = 3y_1 + 2y_2$$

Let $\vec{Y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{Y}'(t) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$

We can write (1) as

$$\vec{Y}'(t) = A\vec{Y}(t) \qquad \text{——— (2)}$$

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Que: How can we solve this system for $\vec{Y}(t)$?

Let us consider the scalar ODE,

$$\frac{dy}{dt} = \lambda y(t) \qquad \text{(3)}$$, λ - growth rate

The solution of this equation is

$$y(t) = C e^{\lambda t}$$

Observe that (2) and (3) are first order linear equation. We expect their solutions to have a similar form.

Take

$$\vec{Y}(t) = C e^{\lambda t} \vec{v} \qquad (4)$$

where λ is a scalar (growth rate)

C — constant

$\vec{v}$ — constant vector

put the solution in (4) into ~~the~~ the system

$$\vec{Y}'(t) = A \vec{Y}(t)$$

$$\lambda \cancel{C} \cancel{e^{\lambda t}} \vec{v} = A \cancel{C} \cancel{e^{\lambda t}} \vec{v}$$

$$\lambda \vec{v} = A \vec{v}$$

$$\boxed{A \vec{v} = \lambda \vec{v}}$$ (eigenvalue problem)

This implies that λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow \quad A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$ (system of algebraic equations)

We seek a nontrivial vector $\vec{v}$ (ie $\vec{v} \neq \vec{0}$) that satisfies this equation.

$\Rightarrow \quad \det(A - \lambda I) = 0$ must hold.

$\therefore$ To get the eigenvalue of matrix A, we solve

$$\boxed{\begin{array}{l} \det(A - \lambda I) = 0 \\ \text{or} \quad |A - \lambda I| = 0 \end{array}}$$

(Characteristic equation)

And for the corresponding eigenvector, we solve

$$\boxed{(A-\lambda I)\vec{v}=\vec{0}}$$

Thus, $\vec{Y}(t)=c\,e^{\lambda t}\,\vec{v}$

is a solution of the system.

If $\vec{Y}_1(t)=c_1 e^{\lambda_1 t}\vec{v}_1$ and

$\vec{Y}_2(t)=c_2 e^{\lambda_2 t}\vec{v}_2$, are solutions

~~$\vec{Y}_1(t)+\vec{Y}_2'$~~ of a linear system, then

$$\vec{Y}_1(t)+\vec{Y}_2(t)=c_1 e^{\lambda_1 t}\vec{v}_1+c_2 e^{\lambda_2 t}\vec{v}_2$$

is also a solution of the system.

This is called <u>principle of superposition</u>.

Let us return to the example

$$\vec{Y}'(t) = A\vec{y}(t)$$

where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Let λ be an eigenvalue of A,

then $$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda_1 = 4 \quad \text{or} \quad \lambda_2 = -1$$