Math215/255 Midterm 1 P

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Question 1

(5 points) Solve the following initial value problems for $y(t)$: $ty' + 2y = \cos(t),\ \ y(\pi) = 0.$

$$
\begin{aligned}
y' + \frac{2}{t}y &= \frac{1}{t}\cos(t) \\
\text{integrating factor is} \qquad h(t) &= e^{\int \frac{2}{t}\,dt} = t^2 \\
\frac{d}{dt}(t^2\, y) &= t\cos(t) \\
\text{integrating,} \qquad t^2 y &= t\sin(t) + \cos(t) + c_1 \\
y(t) &= \frac{1}{t}\sin(t) + \frac{1}{t^2}\cos(t) + \frac{c}{t^2}, \\
y(\pi) = 0 &\Longrightarrow c = 1 \\
y(t) &= \frac{1}{t}\sin(t) + \frac{1}{t^2}\cos(t) + \frac{1}{t^2}.
\end{aligned}
$$

Question 2

(10 points) Consider the following system of first order ODEs

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = 2y_1(t) - 2y_2(t)$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = 2y_1(t) + 2y_2(t)$$

(a) Find the general solution of the system. Convert any complex exponentials in your solutions into " real forms" involving sines and cosines.

$$\text{Let} \qquad A = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

For the eigenvalues, $\lambda^2 - 4\lambda + 8 = 0$ and this gives $\lambda_1 = 2 + 2i$ and $\lambda_2 = 2 - 2i$.

For the eigenvector $\vec{v_1}$,

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \vec{v_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

For the eigenvector $\vec{v_2}$,

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \vec{v_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

We know that the general solution of the system is given by

$$\vec{y}(t) = \mathrm{Re}\left(\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t}\right) + \mathrm{Im}\left(\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t}\right)$$

Consider

$$\left(\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)t}\right) = e^{2t}\begin{pmatrix} i \\ 1 \end{pmatrix}(\cos(2t) + i\sin(2t)) = e^{2t}\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + i\,e^{2t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

Therefore, the general solution is

$$\vec{y}(t) = c_1\,e^{2t}\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2\,e^{2t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

(b) Use the initial conditions $y_1(0) = 1$ and $y_2(0) = 2$ to find the constants in your solution.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Longrightarrow c_1 = 2, \quad c_2 = 1.$$

Therefore,

$$\vec{y}(t) = 2\,e^{2t}\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + \;e^{2t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

Question 3

(7 points) Determine the value of k for which the following equation is exact

$$(y\cos(x) + kxe^y)\,\mathrm{dx} + (sin(x) + x^2e^y - 1)\,\mathrm{dy} = 0.$$

For this equation to be exact, we need $M_y = N_x$.

$$M(x,y) = y\cos(x) + kxe^y \quad \Longrightarrow \quad M_y(x,y) = \cos(x) + kxe^y$$
$$N(x,y) = sin(x) + x^2e^y - 1 \quad \Longrightarrow \quad N_x(x,y) = \cos(x) + 2xe^y$$

Comparing M_y and N_x, we have that $k = 2$.

Therefore,the equation is

$$(y\cos(x) + 2xe^y)\,\mathrm{dx} + (sin(x) + x^2e^y - 1)\,\mathrm{dy} = 0,$$

Let $\Phi(x,y) = C$ be the general solution of this equation. Then

$$\frac{\partial\Phi}{\partial x} = M(x,y) = y\cos(x) + 2xe^y \quad \Longrightarrow \quad \Phi(x,y) = y\,\sin(x) + x^2e^y + \gamma_1(y)$$
$$\frac{\partial\Phi}{\partial y} = N(x,y) = sin(x) + x^2e^y - 1 \quad \Longrightarrow \quad \Phi(x,y) = y\,\sin(x) + x^2e^y - y + \gamma_2(x)$$

Comparing the two $\Phi(x,y)$ functions, we have that $\gamma_1(y) = y$ and $\gamma_2(x) = 0$.

Therefore, the general solution is

$$y\,\sin(x) + x^2e^y - y = C$$

Applying the initial condition $y(\pi) = 0$, we have $C = \pi^2$.

$$y\,\sin(x) + x^2e^y - y = \pi^2.$$

Question 4

(i) C

(ii) $\overrightarrow{y}(t) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \longrightarrow \infty$

(iii) B

Question 5

(12 points) Consider the following ODE

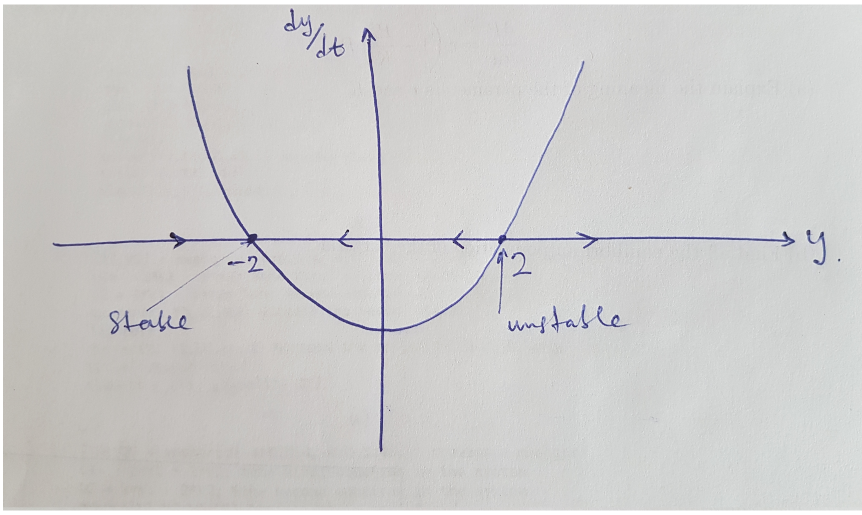
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda(y^2 - 4), \quad \text{where } \lambda > 0\,.$$

(a) Find all the equilibria (steady state solutions) of the differential equation. For steady state solutions, we set $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and solve for y in

$$\lambda(y^2 - 4) = 0,$$

which gives $y = -2$ and $y = 2$.

(b) Sketch the graph of $\frac{\mathrm{d}y}{\mathrm{d}t}$ vs $y(t)$ and use it to determine which of these equlibria is stable, unstable, or semi-stable.



(c)Use the initial condition $y(0) = 1$ to find a solution to the equation.
Hint: You may need the partial fraction $\frac{1}{(y-a)(y-b)} = \frac{A}{(y-a)} + \frac{B}{(y-b)}$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda(y^2 - 4),$$

$$\Longrightarrow \quad \frac{dy}{(y-2)\,(y+2)} = \lambda\, dt$$

$$\frac{1}{(y-2)\,(y+2)} = \frac{A}{(y-2)} + \frac{B}{(y+2)} \quad \Longrightarrow \quad A = \frac{1}{4}\,,\ B = -\frac{1}{4}$$

Therefore, we have

$$\int \left(\frac{1}{4}\frac{1}{(y-2)} - \frac{1}{4}\frac{1}{(y+2)} \right) dP = r\,t\, + c_1\,,$$

$$\frac{1}{4}\ln(y-2) - \frac{1}{4}\ln(y+2) = \lambda\,t\, + c_1\,,$$

$$\frac{(y-2)}{(y+2)} = c_2 e^{4\lambda t}\,, \qquad (c_2 = e^{4\lambda t})$$

$$y(t) = \frac{2 + 2c_2 e^{4\lambda t}}{1 - c_2 e^{4\lambda t}}$$

Applying the initial condition $y(0) = -1$, we have $c_2 = -3$.

$$y(t) = \frac{2 - 6e^{4\lambda t}}{1 + 3e^{4\lambda t}}.$$

(d) Find the limit of this solution $y(t)$ as $t \longrightarrow \infty$.

$$\lim_{t \longrightarrow \infty} y(t) = \frac{2 - 6e^{4\lambda t}}{1 + 3e^{4\lambda t}} = -2.$$