Math 215/255
Midterm 2, Nov 15, 2017

Name: **SID:**

Instructor: **Section:**

Instructions

- The total time allowed is 50 minutes.
- The total score is 40 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators, phones and cheat sheets are not allowed.

Problem	Points	Score
1	14	
2	12	
3	14	
TOTAL	40	

1. (14 points)

a) (5 points) Find a general form of homogeneous solutions for the equation

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}(t).$$

Solution:

$(2-\lambda)(1-\lambda) = 0.$

$\lambda_1 = 2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

$\lambda_2 = 1,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Solution: $\vec{x}_H(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) (2 points) For which initial conditions will the solution remain bounded for large t.

Solution:

Solution: Only $\vec{x}(0) = \vec{0}$.

c) (7 points) Find a particular solution to

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 2t\mathrm{e}^{-t} \\ 2t\mathrm{e}^{-t} \end{bmatrix}.$$

Solution:

$\vec{x}_P(t) = At\mathrm{e}^{-t}\vec{v}_2 + B\mathrm{e}^{-t}\vec{v}_2.$

$$\frac{d}{dt}\vec{x}_P(t) - \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}_P(t) = \left[A\mathrm{e}^{-t} - At\mathrm{e}^{-t} - B\mathrm{e}^{-t} - At\mathrm{e}^{-t} - B\mathrm{e}^{-t}\right]\vec{v}_2 \qquad (1)$$

$A = -1.\ A - 2B = 0,\ B = -\frac{1}{2}.$

Solution: $\vec{x}_P(t) = (-t\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-t}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

2. (12 points) A damped spring with mass 2 kg has friction constant 1 kg/s and spring constant k kg/s^2.

   a) (4 points) For what values of $k \geq 0$ is the spring under-damped, over-damped, and critically-damped?

$$2x''(t) + x'(t) + kx(t) = 0.$$

   $\lambda^2 + \frac{1}{2}\lambda + \frac{k}{2} = 0$, $\lambda = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} - \frac{k}{2}}$

   Solution:
   $k > \frac{1}{2}$ underdamped
   $k = \frac{1}{2}$ critically damped
   $k < \frac{1}{2}$ overdamped.

   b) (5 points) For $k = 1$ a force is applied of $3\sin(2t)$ kg m/s^2, Compute a particular solution of the spring equation.
   $x_P(t) = A\sin(2t) + B\cos(2t)$,

   $2x_P''(t)+x_P'(t)+x_P(t) = -2A\sin(2t)-2B\cos(2t)+A\cos(2t)-B\sin(2t)+A\sin(2t)+B\cos(2t).$

   $-A - B = 3$, $-B + A = 0$, $A = B = -\frac{3}{2}$.

   Solution: $$x_P(t) = -\frac{3}{2}\sin(2t) - \frac{3}{2}\cos(2t)$$

   c) (3 points) If there was no friction, but still mass 2 kg and spring constant $k = 1$, give a forcing term that would exhibit resonance.

   Solution: $F_0\cos(\frac{\sqrt{2}}{2}t)$.

3. (14 points) The following questions concern the equation

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -12 & 8 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

a) (4 points) Find a fundamental matrix for the homogeneous part of the equation.
$(-\lambda)(8-\lambda) - 12 = \lambda^2 - 8\lambda + 12,\ \lambda_1 = 2,\ \lambda_2 = 6$
$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

Solution: $X(t) = \begin{bmatrix} e^{2t} & e^{6t} \\ 2e^{2t} & 6e^{6t} \end{bmatrix}$

b) (6 points) Given the particular solution $\vec{y}_P(t) = \begin{bmatrix} te^{2t} \\ e^{2t} + 2te^{2t} \end{bmatrix}$, find the forcing term $g(t)$.

Solution:
$\frac{d}{dt}\vec{y}_P(t) = \begin{bmatrix} e^{2t} + 2te^{2t} \\ 4e^{2t} + 4te^{2t} \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ -12 & 8 \end{bmatrix} \vec{y}_P(t) = \begin{bmatrix} e^{2t} + 2te^{2t} \\ -12te^{2t} + 8e^{2t} + 16te^{2t} \end{bmatrix}$
$g(t) = -4te^{2t} - 8e^{2t} + 4e^{2t} + 4te^{2t} = -4e^{2t}$

Solution: $g(t) = -4e^{2t}$

c) (4 points) Solve for $\vec{y}(t)$ with the forcing of part b) and the initial conditions

$$\vec{y}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Solution:
$\vec{y}(t) = \begin{bmatrix} te^{2t} - e^{2t} + e^{6t} \\ e^{2t} + 2te^{2t} - 2e^{2t} + 6e^{6t} \end{bmatrix}$

Solution: $\vec{y}(t) = \begin{bmatrix} te^{2t} - e^{2t} + e^{6t} \\ 2te^{2t} - e^{2t} + 6e^{6t} \end{bmatrix}$