# Math 215/255
# Midterm 2, Nov 15, 2017

**Name:** **SID:**

**Instructor:** **Section:**

## Instructions

- The total time allowed is 50 minutes.
- The total score is 40 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators, phones and cheat sheets are not allowed.

| Problem | Points | Score |
|---|---|---|
| 1 | 14 | |
| 2 | 12 | |
| 3 | 14 | |
| **TOTAL** | 40 | |

1. (14 points)

a) (5 points) Find a general form of homogeneous solutions for the equation

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \vec{x}(t).$$

**Solution**:

$(1-\lambda)(2-\lambda) = 0.$

$\lambda_1 = 1,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

$\lambda_2 = 2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

Solution: $\vec{x}_H(t) = c_1 \mathrm{e}^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b) (2 points) For which initial conditions will the solution remain bounded for large $t$.

**Solution**:

Solution: Only $\vec{x}(0) = \vec{0}$.

c) (7 points) Find a particular solution to

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 3t\mathrm{e}^{-t} \\ -3t\mathrm{e}^{-t} \end{bmatrix}.$$

**Solution**:

$\vec{x}_P(t) = At\mathrm{e}^{-t}\vec{v}_2 + B\mathrm{e}^{-t}\vec{v}_2.$

$$\frac{d}{dt}\vec{x}_P(t) - \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \vec{x}_P(t) = \left[A\mathrm{e}^{-t} - At\mathrm{e}^{-t} - B\mathrm{e}^{-t} - 2At\mathrm{e}^{-t} - 2B\mathrm{e}^{-t}\right] \vec{v}_2 \tag{1}$$

$A = -1.\ A - 3B = 0,\ B = -\frac{1}{3}.$

Solution: $\vec{x}_P(t) = (-t\mathrm{e}^t - \frac{1}{3}\mathrm{e}^t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

2. (12 points) A damped spring with mass 4 kg has friction constant 2 kg/s and spring constant $k$ kg/s$^2$.

   a) (4 points) For what values of $k \geq 0$ is the spring under-damped, over-damped, and critically-damped?

$$4x''(t) + 2x'(t) + kx(t) = 0.$$

   $\lambda^2 + \frac{1}{2}\lambda + \frac{k}{4} = 0$, $\lambda = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} - k}$

   Solution:

   $k > \frac{1}{4}$ underdamped
   $k = \frac{1}{4}$ critically damped
   $k < \frac{1}{4}$ overdamped.

   b) (5 points) For $k = 2$ a force is applied of $3\sin(t)$ kg m/s$^2$, Compute a particular solution of the spring equation.
   $x_P(t) = A\sin(t) + B\cos(t)$,

   $4x_P''(t)+2x_P'(t)+2x_P(t) = -4A\sin(t)-4B\cos(t)+2A\cos(t)-2B\sin(t)+2A\sin(t)+2B\cos(t)$.

   $-2A - 2B = 3$, $-2B + 2A = 0$, $A = B = -\frac{3}{4}$.

   Solution:

$$x_P(t) = -\frac{3}{4}\sin(t) - \frac{3}{4}\cos(t)$$

   c) (3 points) If there was no friction, but still mass 4 kg and spring constant $k = 2$, give a forcing term that would exhibit resonance.

   Solution: $F_0 \cos(\frac{\sqrt{2}}{2}t)$.

3. (14 points) The following questions concern the equation

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

a) (4 points) Find a fundamental matrix for the homogeneous part of the equation.

**Solution**:

$(-\lambda)(-8-\lambda) - 12 = \lambda^2 + 8\lambda + 12$, $\lambda_1 = -2$, $\lambda_2 = -6$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Solution: $X(t) = \begin{bmatrix} e^{-2t} & e^{-6t} \\ -2e^{-2t} & -6e^{-6t} \end{bmatrix}$

b) (6 points) Given the particular solution $\vec{y}_P(t) = \begin{bmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \end{bmatrix}$, find the forcing term $f(t)$.

**Solution**:

Solution: $f(t) = 4e^{-2t}$.

c) (4 points) Given the particular solution $\vec{y}_P(t) = \begin{bmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \end{bmatrix}$, find the forcing term $f(t)$.

**Solution**:

Solution: $\vec{y}(t) = \begin{bmatrix} te^{-2t} - e^{-2t} + e^{-6t} \\ -2te^{-2t} + 3e^{-2t} - 6e^{-6t} \end{bmatrix}$.