To compute the eigenvectors, we solve

$$(A - \lambda I)\vec{v} = \vec{0}$$

For $\lambda_1 = 4$,

$$(A - \lambda_1 I)\vec{v} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow -3v_1 + 2v_2 = 0$$

$$v_1 = \tfrac{2}{3} v_2$$

take $v_2 = 3$, $v_1 = 2$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Alternatively, ~~since~~ since we will always end up with a reduced system of the form

$$\left[\begin{array}{cc|c} * & * & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which implies that we have one free variable, we can always take one of the entries of our eigenvector to be $\underline{1}$ and sub. into one of the equations to get the other entry.

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

take $v_2 = 1$ $\qquad -3v_1 + 2 = 0$

$\qquad\qquad v_1 = 2/3$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

$$\therefore \ \vec{Y}_1(t) = c_1 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

is a solution.

For $\lambda_2 = -1$,

$$(A - \lambda_2 I)\vec{V}_2 = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \ \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \ \vec{Y}_2(t) = c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ The general solution of the system is

$$\vec{Y}(t) = \vec{Y}_1(t) + \vec{Y}_2(t) = c_1 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Given an initial condition

$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$

$$c_1 e^{4(0)} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-1(0)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$1 - 3/2$

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 3 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & -5/2 & -1/2 \end{array}\right] R_2 - 3R_1$$

$$-5/2\, c_2 = -1/2 \quad \Rightarrow \quad c_2 = 1/5$$

$$c_1 + \tfrac{1}{2} c_2 = 1/2$$

$$c_1 = \tfrac{1}{2} - \tfrac{1}{2} c_2 = \frac{1}{2} - \frac{1}{10} = \frac{4}{10} = \underline{\underline{2/5}}$$

$$\therefore \vec{Y}(t) = \frac{2}{5} e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{5} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example! solve $y_1'(t) = 2y_1 + y_2$

$y_2'(t) = y_1 + 2y_2$

Let $\vec{y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{y}'(t) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$

$$\vec{y}'(t) = A\vec{y}(t)$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Let λ be an eigenvalue of A.

$$|A - \lambda \underline{I}| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

Alternatively

$\lambda^2 - tr(A)\lambda + det(A) = 0$

trace of A, $tr(A) =$ sum of diagonal entries $= 4$

$det(A) = 3$

$\lambda^2 - 4\lambda + 3 = 0$

For $\lambda_1 = 3$,

$$(A - \lambda_1 I)\vec{V_1} = \vec{0}$$

$$\Rightarrow \left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

take $V_2 = 1$, $\quad -v_1 + v_2 = 0$

$V_1 = V_2 = 1$

$$\vec{V_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$,

$$(A - \lambda_2 I)\vec{V_2} = \vec{0}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{V_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ The general solution is

$$\vec{Y}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Repeated Eigenvalues (defective case)

Example: Solve $\vec{Y}'(t) = A\vec{Y}(t)$

~~wh~~ where $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

Compute the eigenvalues an eigenvectors.

For the eigenvalues,

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1 \text{ (twice)}.$$

For the eigenvector,

$$(A - \lambda I)\vec{V}_1 = \vec{0}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \vec{Y}_1(t) = c_1 e^{t}\begin{pmatrix}1\\1\end{pmatrix}$$

Que.! How do we get $\vec{Y}_2(t)$?

<u>Guess</u>!

$$\vec{Y}_2(t) = c_2(\vec{v}_2 + t\vec{v}_1)e^{\lambda t}.$$

How do we get $\vec{v}_2$?

Sub. $\vec{Y}_2(t)$ into the system.

$$\vec{Y}_2{}'(t) = A\vec{Y}_2(t)$$

$$\cancel{c_2}\lambda\vec{v}_2 e^{\lambda t} + \cancel{c_2}\vec{v}_1 e^{\lambda t} + \cancel{c_2}\lambda t\vec{v}_1 e^{\lambda t} = A\left[\cancel{c_2}(\vec{v}_2 + t\vec{v}_1)e^{\lambda t}\right]$$

$$(\lambda\vec{v}_2 + \vec{v}_1 + \lambda t\vec{v}_1)\cancel{e^{\lambda t}} = A(\vec{v}_2 + t\vec{v}_1)\cancel{e^{\lambda t}}$$

$$\lambda\vec{v}_2 + \vec{v}_1 = A\vec{v}_2 + t\underbrace{(A\vec{v}_1 - \lambda\vec{v}_1)}_{=\vec{0}}$$

(eigenvalue problem for $\lambda_1, \vec{v}_1$)

$$\lambda \vec{V}_2 + \vec{V}_1 = A\vec{V}_2$$

$$A\vec{V}_2 - \lambda \vec{V}_2 = \vec{V}_1$$

$$\boxed{(A - \lambda I)\vec{V}_2 = \vec{V}_1}$$

We can solve this nonhomogeneous system to get $\vec{V}_2$.

To get $\vec{V}_2$, we solve

$$\left(\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad u_1 - u_2 = 1$

$u_1 = 1 + u_2$

take $u_2 = 0$, $\Rightarrow u_1 = 1$

$\therefore \vec{V}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{y}_2(t) = c_2\left[\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right]e^{t}$$

$\Rightarrow$ the general solution is

$$\vec{y}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\left[\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right]e^{t}$$