Webwork problem (Hint)

$$y'' + by' + cy = \alpha \frac{e^{-t}}{t^2+1}$$

Example: $y'' + 8y' + 16y = \dfrac{12e^{-4t}}{t^2+1}$ (1)

Convert to system

Let $y_1 = y \Rightarrow y_1' = y' = y_2$

$y_2 = y_1' \Rightarrow y_2' = y''$

from the ODE

$$y'' = -8y' - 16y + \overbrace{\underbrace{\frac{12e^{-4t}}{t^2+1}}_{f(t)}}^{f(t)}$$

$$y'' = -8y_2 - 16y_1 + f(t)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -16 & -8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

f(t)

So that

$$\vec{g}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

We know that the general Solution is

$$\vec{y}(t) = \underbrace{\Phi \int \Phi^{-1} \vec{g}\, dt}_{y_p} + \underbrace{\Phi \vec{c}}_{y_H}$$

$$y_p = \Phi \int \Phi^{-1} \vec{g}\, dt$$

To construct the fundamental matrix, we solve $\vec{y}\,' = A\vec{y}$

$$\lambda_1 = \lambda_2 = -4$$

For $\vec{v}_1$, , $A = \begin{pmatrix} 0 & 1 \\ -16 & -8 \end{pmatrix}$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{pmatrix} +4 & 1 \\ -16 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

For $\vec{v}_2$, $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{pmatrix} +4 & 1 \\ -16 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$+4u_1 + u_2 = -1$$

$u_2 = 0$, u_1

$$\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{y}_H = c_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t} + c_2 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right) e^{-4t}$$

$$\underline{\Psi} = \begin{pmatrix} -e^{-4t} & -t e^{-4t} \\ 4e^{-4t} & (-1+4t)e^{-4t} \end{pmatrix}$$

We can then find $\vec{y}_p$ using

$$\vec{y}_p = \underline{\Psi} \int \underline{\Psi}^{-1} \vec{g} \, dt$$

where $\vec{g}(t) = \begin{pmatrix} 0 \\ \dfrac{12e^{-4t}}{t^2+1} \end{pmatrix}$

Suppose we got $\vec{y}_p = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

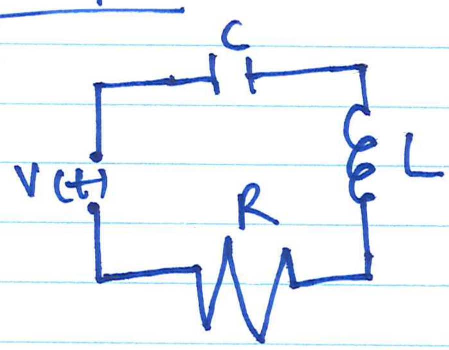
$\therefore$ The y_p for the second order problem in (*) is α_1 (i.e. the first component of $\vec{y}_p$).

Frequency Response

Given a damped oscillator that is forced by an oscillating function, can we compute the following:

* The amplitude of response of the oscillator
* The frequency that gives maximum amplitude of response.

Example: consider the LCR circuit



We know that the charge on capacitor satisfies

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

Since we want a periodic forcing, we set

$$V(t) = \mu\cos(\omega t), \quad \mu > 0 \text{ constant}.$$

We have

$$L\frac{d^2\varphi}{dt^2} + R\frac{d\varphi}{dt} + \frac{1}{C}\varphi = \mu\cos(\omega t)$$

For the homogeneous solution, we solve

$$L\varphi'' + R\varphi' + \frac{1}{C}\varphi = 0$$

$$\Rightarrow \quad L\lambda^2 + R\lambda + \tfrac{1}{C}\varphi = 0$$

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

Suppose $\left(R^2 - 4L/C\right) < 0$, then λ is complex

Let $\lambda = \alpha \pm i\beta$

$$\Rightarrow \quad \alpha = -R/2L \,, \quad \beta = \frac{1}{2L}\sqrt{4L/C - R^2}$$

$\Rightarrow$

$$\varphi_H = c_1 e^{-R/2L\, t}\cos(\beta t) + c_2 e^{-R/2L\, t}\sin(\beta t)$$

For φ_p, consider

$$L\varphi_p'' + R\varphi_p' + \frac{1}{C}\varphi_p = \mu \, Re\left(e^{i\omega t}\right)$$

we solve

$$L\varphi_p'' + R\varphi_p' + 1/C\, \varphi_p = \mu e^{i\omega t} \quad \text{———} \; *1$$

Guess: $\varphi_p = Ae^{i\omega t}$, $\varphi_p' = i\omega A e^{i\omega t}$

$$\varphi_p'' = -\omega^2 A e^{i\omega t}$$

put φ_p into $*1$, we have

$$A = \frac{\mu}{\left[(-L\omega^2 + 1/C) + i\omega R\right]} \times \frac{\left[(-L\omega^2 + 1/C) - i\omega R\right]}{\left[(-L\omega^2 + 1/C) - i\omega R\right]}$$

$$A = \frac{\mu\left[(-L\omega^2 + 1/C) - i\omega R\right]}{(-L\omega^2 + 1/C)^2 + (\omega R)^2}$$

$$\therefore \varphi_p = Ae^{i\omega t} = A\left(\cos(\omega t) + i\sin(\omega t)\right)$$

$$Re(\varphi_p) = \frac{\mu(-L\omega^2 + 1/C)}{(-L\omega^2 + 1/C)^2 + (\omega R)^2}\cos(\omega t) + \frac{\mu\omega R}{(-L\omega^2 + 1/C) + (\omega R)^2}\sin(\omega t)$$

Let $D(\omega) = \dfrac{\mu\left(-L\omega^2 + 1/c\right)}{\left(-L\omega^2 + 1/c\right)^2 + (\omega R)^2}$

$$E(\omega) = \frac{\mu \omega R}{\left(-L\omega^2 + 1/c\right) + (\omega R)^2}$$

$$\mathrm{Re}(\varphi_p) = D(\omega)\cos(\omega t) + E(\omega)\sin(\omega t)$$

Recall that we can write this solution in the form

$$\mathrm{Re}(\varphi_p) = H(\omega)\cos(\omega t + \gamma)$$

where $H(\omega) = \sqrt{D^2 + E^2}$ (amplitude)

and $\gamma = \arctan\left(E/D\right)$

$\therefore$ The general solution is

$$\varphi(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t) + H\cos(\omega t + \gamma)$$

Since $\alpha < 0$, as $t \to \infty$

$$\varphi(t) \approx H\cos(\omega t + \gamma)$$

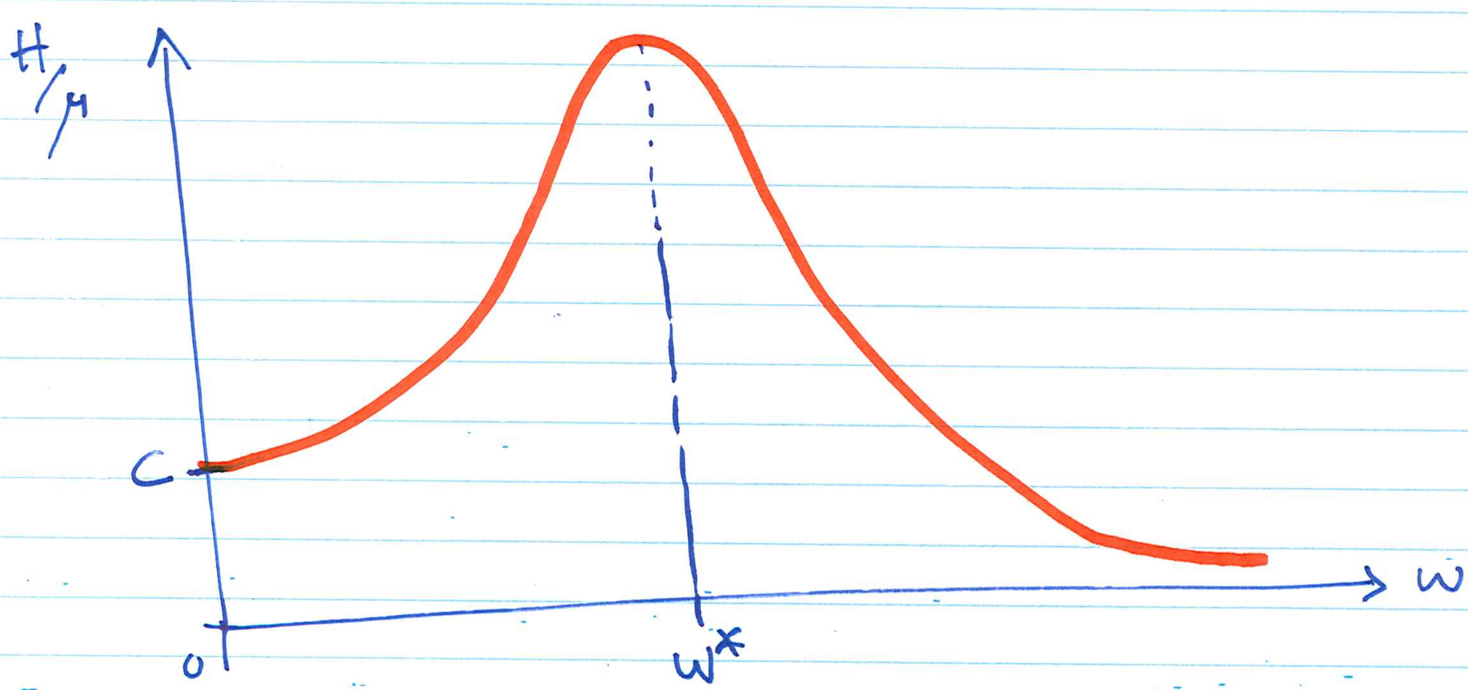
This means that after a long time the system will oscillate with amplitude $H(\omega)$ and frequency ω.

consider the amplitude

$$H(\omega) = \sqrt{D^2 + E^2}$$

Sub. in $D(\omega)$ and $E(\omega)$, we have

$$H(\omega) = \frac{A}{\sqrt{(-L\omega^2 + 1/C)^2 + (\omega R)^2}}$$



ω^* is the frequency that gives maximum amplitude.

To compute w^*, we solve for w in the equation

$$\frac{d H^2}{dw} = 0$$

and this gives

$$w^* = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

If the resistance, R is small, that is as $R \longrightarrow 0$,

$$w^* \approx \sqrt{1/LC}$$ (natural frequency of undamped system)

Nonlinear systems

consider the system

$$\vec{y}' = \vec{F}(t, \vec{y}) \quad \text{——— (1)}$$

* if $\vec{F}(t, \vec{y})$ is a nonlinear function, then the system is nonlinear.

* If $\vec{F} \equiv \vec{F}(\vec{y})$ and nonlinear, then the system is autonomous nonlinear.