ORDER of an ODE, is the highest number of derivatives in the equation.

Examples:

(i) $y'' - 4y' + 4y = 0$ — 2nd order

(ii) $y' + \sqrt{y}\, y'''' = \sin(t)$ — 4th order

Linear / Nonlinear ODEs

If an ODE contains only linear functions of $y, y', y'', \ldots$ then it is linear

More precisely,

$a(t)\, dy/dt + b(t)\, y = c(t)$ — 1st order linear

$$a(t) \frac{d^2y}{dt^2} + b(t) \frac{dy}{dt} + c(t)\, y = d(t)$$

— 2nd order linear

Example:

(i) $x^2 y'' + x y' + y = 0$ — linear

(ii) $y y' = 1$ — Nonlinear

(iii) $t^2 y'' - \sin(t) y' = e^t$ — linear

(iv) $y'' + \sin(y) = 0$ — Nonlinear

Linear ODE: – usually have solution
– there are techniques for solving them exactly/analytically.

Nonlinear ODE: – May not have solution or have unique solution.
– ~~May not~~ We may not be able to solve analytically.
– usually they are solved numerically or by approximation

Example: Solve the IVP (Initial Value problem)

$$\frac{dy}{dt} = e^{-2t}, \quad y(0) = 5, \quad t > 0$$

$$\frac{dy}{dt} = e^{-2t}$$

$$\frac{dy}{\cancel{dt}} \cdot \cancel{dt} = e^{-2t}\, dt$$

$$\int dy = \int e^{-2t}\, dt + c$$

$$y = \frac{e^{-2t}}{-2} + c \text{ — general solution}$$

Apply the initial condition $y(0) = 5$

$$5 = \frac{e^{-2(0)}}{-2} + c \Rightarrow c = 5 + \frac{1}{2} = \frac{11}{2}$$

$$\therefore y = -\frac{1}{2}e^{-2t} + \frac{11}{2}$$

Let us solve the same problem in a slightly different way (using definite integral)

We have

$$dy/dt = e^{-2t}$$

$$\int_{y_0}^{y} dz = \int_{t_0}^{t} e^{-2u}\, du$$

from I.C. $y(0) = 5 \Rightarrow t_0 = 0, \; y_0 = 5$

$$\int_5^y 1\, dz = \int_0^t e^{-2u}\, du$$

$$z\Big|_5^y = -\frac{1}{2} e^{-2u}\Big|_0^t$$

$$y - 5 = -\frac{1}{2} e^{-2t} + \frac{1}{2}$$

$$y = 11/2 - \frac{1}{2} e^{-2t}$$

Example: Solve the IVP

$$dy/dt = e^{-t^2}, \quad y(0) = 1/2$$

$$dy/dt = e^{-t^2}$$

$$\cancel{\int} dy = e^{-t^2}\,dt$$

$$\int_{y_0}^{y} 1\,dz = \int_{t_0}^{t} e^{-u^2}\,du$$

$$\cancel{z\Big|_{y_0}^{y}} \qquad y_0 = 1/2, \quad t_0 = 0$$

$$\int_{1/2}^{y} 1\,dz = \int_{0}^{t} e^{-u^2}\,du$$

$$z\Big|_{1/2}^{y} = \int_{0}^{t} e^{-u^2}\,du$$

$$y = \int_0^t e^{-u^2}\,du + \frac{1}{2}$$

The integral on the R.H.S does not have a close form. Therefore, we can write the solution of our equation as a definite integral.

EXISTENCE & UNIQUENESS OF SOLUTIONS OF ODEs

Given an I.V.P

$$y' = f(x, y), \quad y(x_0) = y_0$$

fundamental questions to ask:

(i) Does a solution exist?

(ii) If a solution exists, is the solution unique?

Example: Solve

$$xy' = 4, \quad y(0) = 5$$

try to solve this ...

Solution : $y = 4 \ln |x| + C$

But $\ln(0) = -\infty$,

$\therefore$ the IVP does not have a solution that satisfies the initial condition

$\Rightarrow$ The equation does not have a ~~smooth solution~~ continuous solution near $x = 0$.

Example: ~~solve~~ Solve

$$dy/dx = 3y^{2/3}, \quad y(2) = 0$$

$$\int y^{-2/3}\, dy = \int 3\, dx + C_1$$

$$3y^{1/3} = 3x + C_1$$

$$y^{1/3} = x + C_2 \qquad \left(C_2 = C_1/3 \right)$$

$y(2) = 0$

$$\Rightarrow \quad 0 = 2 + C_2$$

$$C_2 = -2$$

$$y^{1/3} = x - 2$$

Observe that if $y = 0$,

$$dy/dx = 0, \quad y(2) = 0$$

$\therefore$ $y = 0$ is also a solution of the IVP

$\Rightarrow$ The IVP does not have a unique solution close to $(2, 0)$.

Picard's theorem (existence & uniqueness of solutions of ODEs)

Given an IVP

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

If $f(x,y)$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous near some point (x_0, y_0), the $\exists$ (there exist) a solution to the IVP ~~nee~~ near (x_0, y_0) and the solution is unique near this point.

Check:

(i) $xy' = 4$, $y(0) = 5$

$y' = \cancel{4x}\ 4/x$

$\Rightarrow f(x,y) = 4/x$

At $x = 0$, $f(x,y)$ is not defined. $\therefore$ this violate's the assumptions ~~theorem~~ of the theorem

(11) $\frac{dy}{dx} = 3y^{2/3}$

$f(x,y) = 3y^{2/3}$ — f is continuous

$\frac{\partial f}{\partial y} = 2y^{-1/3} = \frac{2}{y^{1/3}}$

At $y=0$, $\frac{\partial f}{\partial y}$ does not exist.

This violates the assumptions of the theorem.

Note that $f(x,y)$ is not continuous or $\frac{\partial f}{\partial y}$ is not continuous does not necessarily mean that ~~the~~ an IVP does not have a solution or a solution is not unique.

Also, the statement of the theorem is not ~~iff~~ if and only if.