MIDTERM 1 REVIEW

Example

Solve
$$\vec{x}'(t) = \underbrace{\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}}_{A} \vec{x}(t)$$

For the eigenvalues

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = 1 \quad (\text{twice}).$$

For $\vec{v}_1$, $\quad (A - \lambda I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\vec{v}_2$, $\quad (A - \lambda I)\vec{v}_2 = \vec{v}_1$ ✓

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$u_1 - 2u_2 = 1$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general solution has the form,

$$\vec{Y}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2\left[\vec{v}_2 + t\vec{v}_1\right]e^{\lambda t}$$

$$= c_1 e^{t}\begin{pmatrix}2\\1\end{pmatrix} + c_2\left[\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}\right]e^{t}.$$

Scalar ODE

$$y'' - 2y' + y = 0$$

$$y = e^{\lambda t}, \quad y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}$$

$$\lambda^2 \cancel{e^{\lambda t}} - 2\lambda \cancel{e^{\lambda t}} + \cancel{e^{\lambda t}} = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = 1 \quad (\text{twice})$$

$$y_1 = e^{\lambda t} = e^{t}$$

$$y_2 = te^{\lambda t} = te^{t}.$$

Let check if y_2 satisfies the ODE.

$$y_2 = te^t \quad , \quad y_2' = e^t + te^t$$

$$y_2'' = 2e^t + te^t$$

$$(2e^t + te^t) - 2(e^t + te^t) + te^t$$

$$= 2e^t + te^t - 2e^t - 2te^t + te^t = 0$$

$\therefore$ y_2 is a solution.

~~False~~ For the vector case, take

$$\vec{y}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 t e^{\lambda t} \vec{v}_2$$

sub. this solution into

$$\vec{y}'(t) = A\vec{y}(t)$$

you will get

$$\vec{v}_2 = t\vec{v}_1$$

which means $\vec{v}_1$ and $\vec{v}_2$ are scalar multiple of each other. Not what we want!!

(4) Example,

Is this equation exact?

$$\left(y e^{2xy} + x\right) dx + x e^{2xy} \, dy = 0$$

The general solution is

$$\frac{1}{2} e^{2xy} + x^2/2 = C$$

<u>Example!</u> Solve $y' + 2y = t e^{-3t}$

$$y(t) = -t e^{-3t} - e^{-3t} + c_1 e^{-2t}$$