Last class

* We started looking at Epidemic models.
* Model of HIV in a population.
* Found the steady-states and contracted the Jacobian matrix.
* For the disease free equilibrium $(b/d, 0)$, we got

$$\vec{DF} = \begin{pmatrix} -d & -\beta \\ 0 & \beta-(\alpha+d) \end{pmatrix}$$

Eigenvalues are

$$\lambda_1 = -d < 0$$

$$\lambda_2 = \beta-(\alpha+d)$$

For the equilibrium to be stable, we need

~~$\lambda_2 = 0$~~ $\lambda_2 < 0$

$$\Rightarrow \quad \beta - (\alpha + d) < 0$$

$$\Rightarrow \quad \frac{\beta}{\alpha + d} < 1$$

let $R_0 = \dfrac{\beta}{\alpha + d}$

Then for $(b/d, 0)$ to be stable we need $R_0 < 1$.

This means that for the disease to be eradicated from the population, we need $\boxed{R_0 < 1}$.

Suppose $\lambda_2 > 0$, then $(b/d, 0)$ is unstable.

$$\therefore \quad \beta - (\alpha + d) > 0$$

$$\Rightarrow \quad \frac{\beta}{\alpha + d} > 1$$

$$\cdot \quad R_0 > 1$$

This means that if $\boxed{R_0 > 1}$, HIV will spread through out the population.

<u>Near the endemic equilibrium, (S_2, I_2)</u>

The Jacobian matrix is

$$D\vec{F}(S_2, I_2) = \begin{pmatrix} -\dfrac{(\beta-(\alpha+d))^2}{\beta} - \beta d & -\dfrac{(\alpha+d)^2}{\beta} \\[2em] \dfrac{(\beta-(\alpha+d))^2}{\beta} & \dfrac{(\alpha+d)(\alpha+d-\beta)}{\beta} \end{pmatrix}$$

Let λ be an eigenvalue of $D\vec{F}(S_2, I_2)$

$$\Rightarrow \quad \lambda^2 - tr(D\vec{F})\lambda + det(D\vec{F}) = 0$$

$\Rightarrow$

$$\lambda_{1/2} = \frac{tr(D\vec{F})}{2} \pm \frac{1}{2}\sqrt{\left(tr(D\vec{F})\right)^2 - 4\det(D\vec{F})}$$

We shall use the ~~the~~ trace and determinant of $D\vec{F}$ to ~~the~~ determine the stability of (S_2, I_2).

We know that

$$tr(D\vec{F}) = \lambda_1 + \lambda_2$$

$$\det(D\vec{F}) = \lambda_1 \lambda_2$$

We have

$$\det(D\vec{F}) = \frac{1}{\beta}(\alpha+d)\left(\beta-(\alpha+d)\right)(\beta-\alpha)$$

Recall that $\beta-(\alpha+d) > 0$ and $\beta-\alpha > 0$ must hold for this equilibrium to exist.

$\Rightarrow \det(D\vec{F}) > 0$

$$tr(D\vec{F}) = -(\beta-(\alpha+d))(\beta+d) < 0$$

using the (det, tr)-plane chart (at end of note),

we can conclude that (S_2, I_2) is

* stable since $\boxed{\det(D\vec{F}) > 0 \text{ and } tr(DF) < 0}$

Recall, for this steady-state to exist, we need $\beta-(\alpha+d) > 0$

$$\Rightarrow \frac{\beta}{\alpha+d} > 1$$

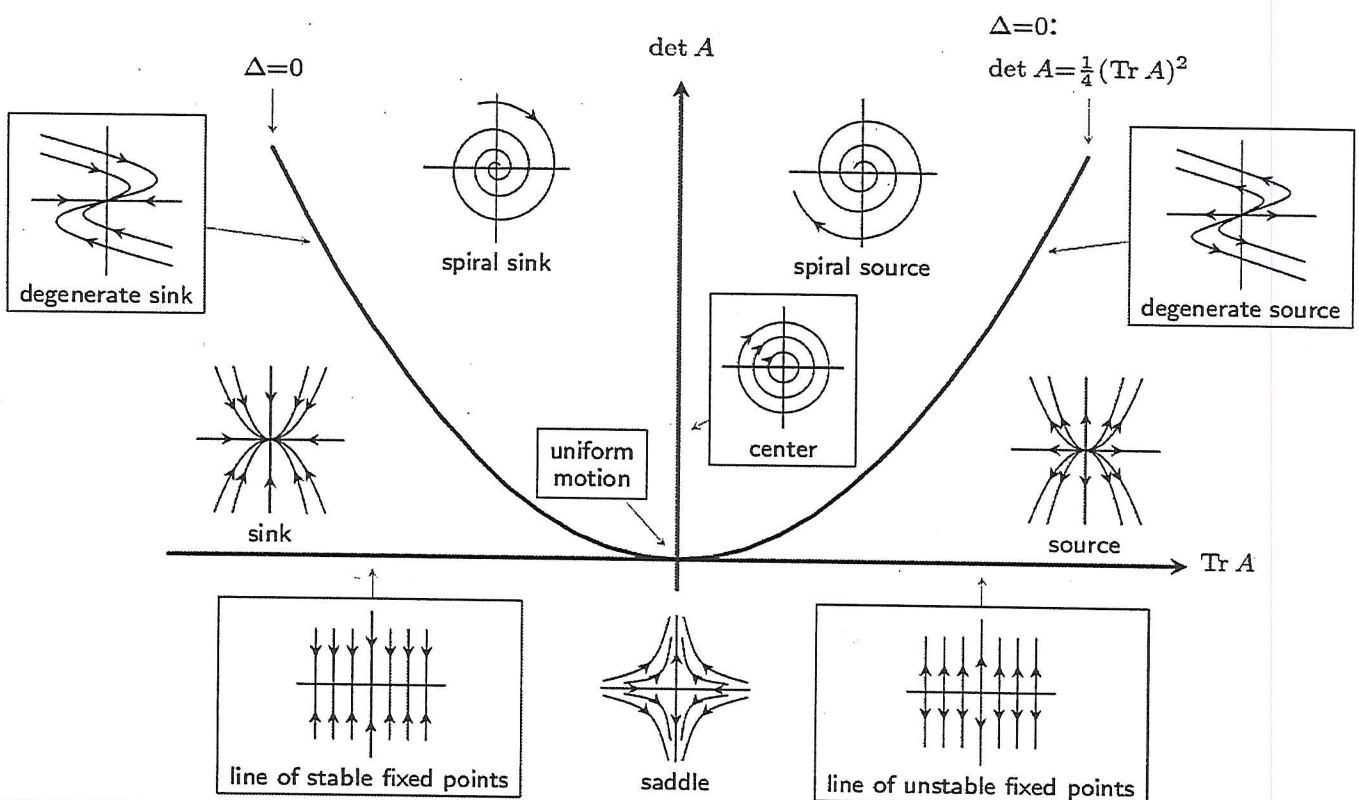
But $R_0 = \dfrac{\beta}{\alpha+d}$

$\Rightarrow$ We need $R_0 > 1$ for the equilibrium to exist and if it exists, it's stable.

$\Rightarrow$ If $\boxed{R_0 > 1}$, the HIV will be endemic in the population.

* You can download the Matlab code and play with the parameters *

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \operatorname{Tr} A)$-plane



Ref: https://tex.stackexchange.com/questions/347201/drawing-the-trace-determinant-diagram-on-latex.

Laplace Transform

Laplace transform is an important tool in mathematics. It involves transforming problems from time-domain to frequency domain.

* When applied to an ODE problem, it transforms the problem into an algebraic equation (which is easier to solve)
* It is convenient for solving problems with step-wise forcing (like in HW 4).

Definition: The L.T. of a function $y(t)$ is given by

$$L[y(t)] = Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

where $s > 0$ is the frequency parameter.

<u>Examples:</u> Find the L.T. of the following functions.

① $y(t) = K$, (k, constant)

$$L[y(t)] = Y(s) = \int_0^{\infty} y(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} K e^{-st}\, dt = K \int_0^{\infty} e^{-st}\, dt$$

$$= K \lim_{A \to \infty} \int_0^{A} e^{-st}\, dt$$

$$= K \lim_{A \to \infty} \left[\frac{e^{-st}}{-s} \right]_0^A$$

$$= K \lim_{A \to \infty} \left[\frac{e^{-sA}}{-s} - \frac{1}{-s} \right]$$

$$Y(s) = K\left(0 + \frac{1}{s}\right) = \frac{k}{s}.$$

② $y(t) = e^{at}, \quad t \geq 0$

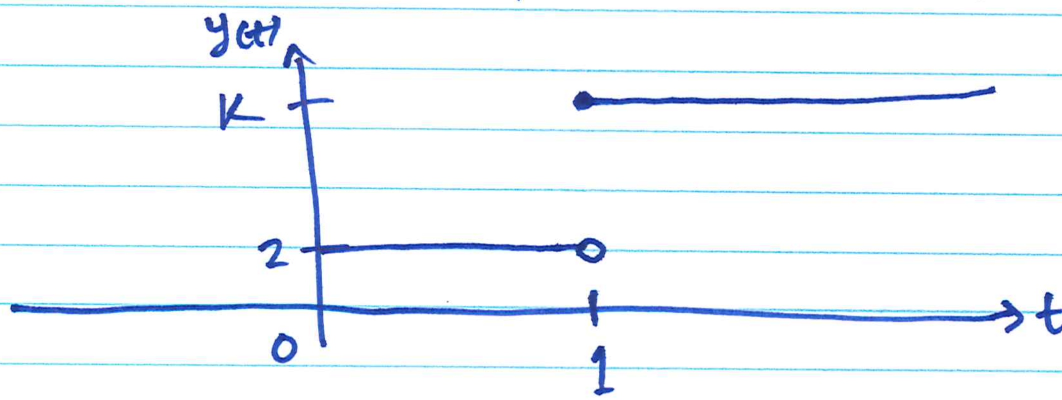
$$L[y(t)] = \int_0^\infty e^{at} \cdot e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)} \right]_0^\infty$$

$$= -\frac{1}{(a-s)}, \quad (a < s)$$

$$\therefore Y(s) = \frac{1}{s-a}, \quad (s > a)$$

③ $y(t) = \begin{cases} 2, & 0 \leq t < 1 \\ K, & t \geq 1 \end{cases}$ (where $K > 2$)



$$L[y(t)] = \int_0^\infty y(t) e^{-st}\, dt$$

$$= \int_0^1 y(t) e^{-st}\, dt + \int_1^\infty y\, e^{-st}\, dt$$

$$= \int_0^1 2\, e^{-st}\, dt + \int_1^\infty K\, e^{-st}\, dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + K\left[\frac{e^{-st}}{-s}\right]_1^\infty$$

$$= \frac{2}{-s}\left(e^{-s} - 1\right) + K\left(-\frac{e^{-s}}{-s}\right)$$

$$= \frac{2}{-s}\left(e^{-s} - 1\right) + K\,\frac{e^{-s}}{s}$$

$$Y(s) = \frac{2}{s} + \frac{1}{s}(K-2)\, e^{-s}, \quad s > 0.$$

④ $y(t) = e^{iat}$

$$L[y] = \int_0^\infty e^{iat} e^{-st}\, dt$$

$$= \int_0^\infty e^{(-s+ia)t}\, dt = \left. \frac{e^{(-s+ia)t}}{(-s+ia)} \right|_0^\infty$$

$$= -\frac{1}{(-s+ia)} \times \frac{(-s-ia)}{(-s-ia)}$$

$$Y(s) = \frac{s+ia}{s^2+a^2} = \boxed{\frac{s}{s^2+a^2}} + i \boxed{\frac{a}{s^2+a^2}}$$

Recall, $e^{iat} = \cos(at) + i\sin(at)$

$$L[e^{iat}] = L[\cos(at)] + i L[\sin(at)]$$

$$\Rightarrow L[\cos(at)] = \frac{s}{s^2+a^2}$$

$$L[\sin(at)] = \frac{a}{s^2+a^2}$$

In general,

$$\boxed{L[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 L[y_1] + \alpha_2 L[y_2]}$$

which means that Laplace transform is a <u>linear transformation</u>.