FORCED SYSTEM

A forced system is of the form

$$\vec{y}'(t) = A\vec{y}(t) + \vec{g}(t) \quad \text{——— } (1)$$

where $\vec{g}(t)$ is the forcing function.

We know that the system (1) is nonhomogeneous and the solution is of the form

$$\vec{y}(t) = \vec{y}_H(t) + \vec{y}_p(t)$$

where

$\vec{y}_H$ is called the homogeneous solution and it satisfies $\vec{y}_H' = A\vec{y}_H$

and $\vec{y}_p$ is the particular solution which satisfies

$$\vec{y}_p' = A\vec{y}_p + \vec{g}(t).$$

Que: How do we find $\vec{Y}_p$?

OR How do we solve the system ① ?

Method I : ~~Cl~~ Undetermined Coefficent (guess and check !!)

Example: Solve

$$y_1'(t) = y_1 + y_2 + 2e^t$$
$$y_2'(t) = 4y_1 + y_2 - e^t$$

Let $\vec{Y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Then, we have

$$\vec{Y}'(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{A} \vec{Y} + \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t}_{\vec{g}(t)}$$

The general solution is

$$\vec{Y}(t) = \vec{Y}_H + \vec{Y}_p$$

To get $\vec{Y}_H$, we solve

$$\vec{Y}'(t) = A\vec{Y}(t)$$

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

For the eigenvalues of A,

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -1$$

For $\lambda_1 = 3$,

$$(A - \lambda_1 I)\vec{V}_1 = \vec{0}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -1$,

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{V}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\therefore$ ~~The~~

$$\vec{Y}_H = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-t}.$$

For the particular solution, let us guess

$$\vec{y}_p = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} e^t \qquad , \quad \vec{g} = \begin{pmatrix}2\\ -1\end{pmatrix} e^t$$

α_1 and α_2 are constants to be determined.

put $\vec{y}_p$ into the forced system.

$\Rightarrow$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} e^t = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix} \cdot \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} e^t + \begin{pmatrix}2\\ -1\end{pmatrix} e^t$$

Divide through by e^t

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} + \begin{pmatrix}2\\ -1\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}\alpha_1 + \alpha_2\\ 4\alpha_1 + \alpha_2\end{pmatrix} + \begin{pmatrix}2\\ -1\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - \begin{pmatrix}\alpha_1 + \alpha_2\\ 4\alpha_1 + \alpha_2\end{pmatrix} = \begin{pmatrix}2\\ -1\end{pmatrix}$$

$$\begin{pmatrix}-\alpha_2\\ -4\alpha_1\end{pmatrix} = \begin{pmatrix}2\\ -1\end{pmatrix}, \quad \Rightarrow \quad \begin{aligned}\alpha_1 &= 1/4\\ \alpha_2 &= -2\end{aligned}$$

$$\therefore \quad \vec{Y}_p = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}.$$

$\therefore$ the general solution is

$$\vec{Y}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

* Suppose we are given an I.C., this is the stage we apply it to get the constants c_1 and c_2.

Remark

* Finding a good/useful guess may be challenging.

Method 2 : Variation of parameters.

Given

$$\vec{y}'(t) = A\vec{y}(t) + \vec{g}(t)$$

Recall, that for a homogeneous system

$$\vec{y}'(t) = A\vec{y}(t)$$

we can write the solution in the form

$$\vec{y}(t) = \Psi \vec{c} \qquad (*)$$

Ψ: fundamental matrix

$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Since the solution of the homogeneous system has the form in $(*)$, it is logical to guess that the forced system has a solution of the form

$$\vec{y}(t) = \Psi \vec{f}(t) \qquad (*1)$$

Que: How do we find $\vec{f}(t)$?

Sub. (*) into ~~to~~ the forced system

to get $\vec{f}(t)$.

$$\vec{y}\,'(t) = A\vec{y}(t) + \vec{g}(t).$$

But our guessed solution is

$$\vec{y}(t) = \Psi(t)\,\vec{f}(t)$$

$$\Rightarrow \vec{y}\,'(t) = \Psi'\vec{f} + \Psi\vec{f}\,' \qquad \text{(product rule)}$$

$\Rightarrow$

$$\Psi'\vec{f} + \Psi\vec{f}\,' = A\Psi\vec{f} + \vec{g}(t)$$

$$\left(\Psi' - A\Psi\right)\vec{f} + \Psi\vec{f}\,' = \vec{g}(t)$$

Since the fundamental matrix satisfies the homogeneous system,

$$\Psi' = A\Psi \Rightarrow \Psi' - A\Psi = 0$$

$$\Rightarrow \quad \Psi\vec{f}\,' = \vec{g}$$

Multiply through by Ψ^{-1} from the left.

$$\vec{f}\,' = \Psi^{-1}\vec{g}$$

$$\Rightarrow \quad \frac{d}{dt}\vec{f} = \Psi^{-1}\vec{g}$$

$$d\vec{f} = \Psi^{-1}\vec{g}\,dt$$

$$\vec{f} = \int \Psi^{-1}\vec{g}\,dt + \vec{c}$$

put $\vec{f}$ into $*_1$

$$\vec{y}(t) = \Psi\vec{f} = \Psi\left(\int \Psi^{-1}\vec{g}\,dt + \vec{c}\right)$$

$$\boxed{\vec{y}(t) = \underbrace{\Psi\int \Psi^{-1}\vec{g}\,dt}_{\text{particular solution}} + \underbrace{\Psi\vec{c}}_{\text{homogeneous solution}}}$$

This is the general solution of the forced system.

Let us apply this formular to our example.

We have

$$\vec{Y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{Y}(t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$$

Recall that:

$$\vec{Y}_H = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\Rightarrow$

$$\Psi = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix}$$

$$\Psi^{-1} = \frac{1}{\det(\Psi)} \begin{pmatrix} 2e^{-t} & e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix}$$

$$\det(\Psi) = 2e^{2t} + 2e^{2t} = 4e^{2t}$$

$\Rightarrow$

$$\Psi^{-1} = \frac{1}{4e^{2t}} \begin{pmatrix} 2e^{-t} & e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 2e^{-3t} & e^{-3t} \\ -2e^{t} & e^{t} \end{pmatrix}$$

$$\Psi^{-1}\vec{g} = \frac{1}{4} \begin{pmatrix} 2e^{-3t} & e^{-3t} \\ -2e^{t} & e^{t} \end{pmatrix} \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 4e^{-2t} - e^{-2t} \\ -4e^{2t} - e^{2t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3e^{-2t} \\ -5e^{2t} \end{pmatrix}$$

$$\Psi^{-1}\vec{g} = \frac{1}{4} \begin{pmatrix} 3e^{-2t} \\ -5e^{2t} \end{pmatrix}$$

$$\int \underline{\Psi}^{-1} \vec{g}\, dt = \frac{1}{4} \int \begin{pmatrix} 3e^{-2t} \\ -5e^{2t} \end{pmatrix} dt$$

$$= \frac{1}{4} \begin{pmatrix} -\frac{3}{2} e^{-2t} \\ -\frac{5}{2} e^{2t} \end{pmatrix}$$

$$\underline{\Psi} \int \underline{\Psi}^{-1} \vec{g}\, dt = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} -\frac{3}{2} e^{-2t} \\ -\frac{5}{2} e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} e^{t} \\ -2 e^{t} \end{pmatrix}$$

** continuation from class **

But $\vec{Y}(t) = \underline{\Psi} \int \underline{\Psi}^{-1} \vec{g}\, dt + \underline{\Psi}\, \vec{c}$

$\therefore$ The general solution is

$$\vec{Y}(t) = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} e^{t} \\ -2e^{t} \end{pmatrix}$$

$\Rightarrow$

$$\vec{Y}(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t}$$

Observe that this solution is the same as what we obtained using the method of undetermined coefficient.