Last class

We started solving the IVP

$$y'' + 6y' + 34y = 30\sin(2t)$$
$$y(0) = 0, \quad y'(0) = 0.$$

After taking the L.T. of the equation, we got

$$L[y] = \frac{60}{(s^2+4)(s^2+6s+34)}$$

To find the inverse L.T. we decompose this fraction using partial fractions. (*)

Let

$$\frac{60}{(s^2+4)(s^2+6s+34)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s+34}$$

and this gives us

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 6 & 1 & 0 & 1 & 0 \\ 34 & 6 & 4 & 0 & 0 \\ 0 & 34 & 0 & 4 & 60 \end{array}\right)$$

with solution vector $\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$

Solving this ~~the~~ system, we got

$$A = -10/29, \quad B = 50/29, \quad C = D = 10/29$$

sub. the result into (*1),

$$L[y] = \frac{\left(-10/29\right)s + \left(50/29\right)}{s^2+4} + \frac{\left(10/29\right)s + \left(10/29\right)}{s^2+6s+34}$$

$$L[y] = -\frac{10}{29}\left[\frac{s-5}{s^2+4}\right] + \frac{10}{29}\left[\frac{s+1}{s^2+6s+34}\right]$$

(*2)

~~cons~~ consider

$$\frac{s-5}{s^2+4} = \frac{s}{s^2+4} - \frac{5}{s^2+4}$$

$$L[\cos(2t)] = \frac{s}{s^2+4}$$

$$\frac{5}{2} \cdot \frac{2}{s^2+4}$$

$$L[\sin(2t)] = \frac{2}{s^2+4}$$

$$L^{-1}\left[\frac{s-5}{s^2+4}\right] = \cos(2t) - \frac{5}{2}\sin(2t)$$ *3

consider

$$\frac{s+1}{s^2+6s+34}$$ *1

use completing the square for the denominator

$$s^2+6s+34 = s^2+6s+9-9+34$$

$$= (s^2+6s+9)+25$$

$$= (s+3)^2+25$$

Therefore,

$$\frac{s+1}{s^2+6s+34} = \frac{s+1}{(s+3)^2+25} = \frac{(s+1)+(2-2)}{(s+3)^2+25}$$

$$= \frac{s+3}{(s+3)^2+25} - \frac{2}{(s+3)^2+25}$$

$\therefore$

$$L^{-1}\left[\frac{s+1}{s^2+6s+34}\right] = L^{-1}\left[\frac{s+3}{(s+3)^2+25}\right] - \frac{2}{5}L^{-1}\left[\frac{5}{(s+3)^2+25}\right]$$

(using formula 15 and 16)

$$\downarrow \quad = e^{-3t}\cos(5t) - \frac{2}{5}e^{-3t}\sin(5t) \text{ ——— } (*4)$$

From $(*2)$, we have

$$y(t) = -\frac{10}{29}L^{-1}\left[\frac{s-5}{s^2+4}\right] + \frac{10}{29}L^{-1}\left[\frac{s+1}{s^2+6s+34}\right]$$

substituting $(*3)$ and $(*4)$ into ~~in~~ this equation,

$$y(t) = -\frac{10}{29}\left[\cos(2t) - \frac{5}{2}\sin(2t)\right] + \frac{10}{29}\left[e^{-3t}\cos(5t) - \frac{2}{5}e^{-3t}\sin(5t)\right]$$

Impulse function

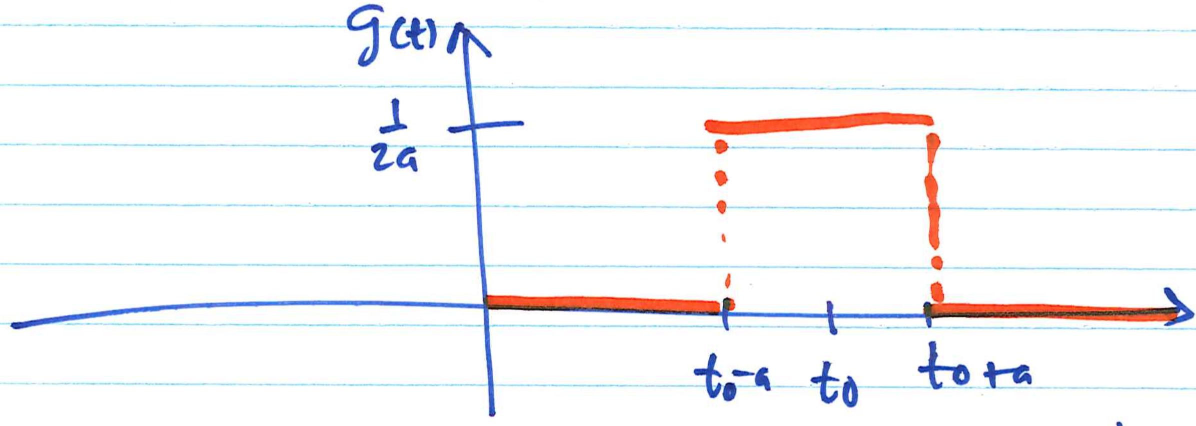
Suppose we have a system that requires an input force of large magnitude over a ~~very~~ short time interval, we can write the ODE for this system using an ~~step~~ impulse function as the forcing function.

For instance: consider

$$a_1 y'' + a_2 y' + a_3 y = g(t)$$

1 (a_1, a_2, a_3 are constants)

where $g(t)$ is given by



assume $a \ll 1$ (very small) $\Rightarrow \frac{1}{2a}$ is very large.

$g(t)$ is an impulse function.

To ~~measure~~ measure the strength of the forcing, we integrate $g(t)$,

$$I = \int_{-\infty}^{\infty} g(t)\, dt = \int_{t_0 - a}^{t_0 + a} g(t)\, dt$$

By definition, (1*)

$$g(t) = \begin{cases} 0, & t \le t_0 - a \\ \frac{1}{2a}, & t_0 - a < t < t_0 + a \\ 0, & t \geq t_0 + a \end{cases}$$

Take $t_0 = 0$, then

$$g(t) = \begin{cases} 0, & t \le -a \\ \frac{1}{2a}, & -a < t < a \\ 0, & t \geq a \end{cases}$$

$$I = \int_{-\infty}^{\infty} g(t)\, dt = \int_{-a}^{a} \frac{1}{2a}\, dt = 1$$

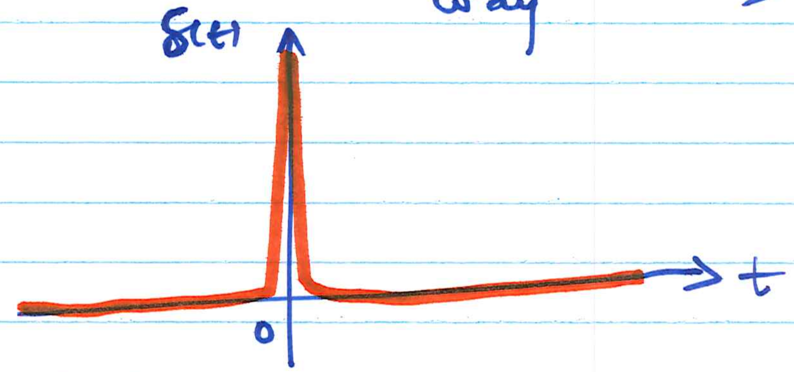
Definition: Dirac delta "function"

$$\delta(t) = \lim_{a \to 0} g(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

(not ~~used~~ usually used this way)

with

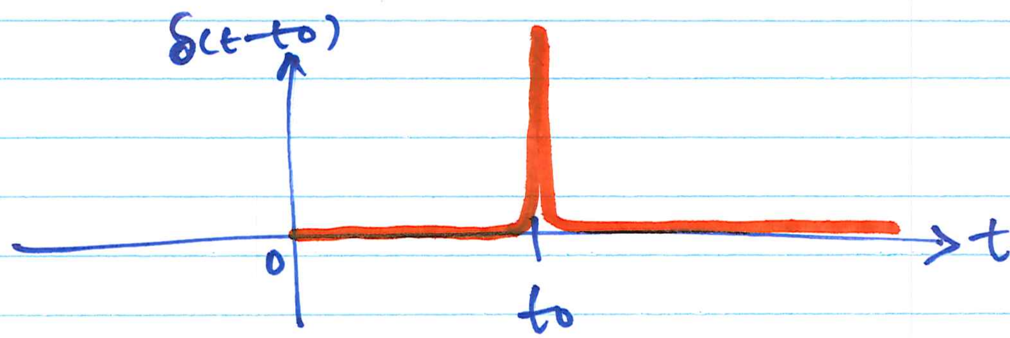
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



shifted Dirac delta function,

$$\delta(t - t_0) = \cancel{\lim} \begin{cases} +\infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$



~~Def:~~ observe

$$\cancel{\int \delta(t-a)} \quad \int_{t_1}^{t_2} \delta(t-a)\, dt = \begin{cases} 1, & a \in (t_1, t_2) \\ 0, & \text{otherwise} \end{cases}$$

Let $f(t)$ be another function,

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \int_{-\infty}^{\infty} \left(\lim_{a \to 0} g(t) \right) f(t)\, dt$$

$$= \lim_{a \to 0} \int_{-\infty}^{\infty} g(t) f(t)\, dt$$

$$= \lim_{a \to 0} \int_{t_0-a}^{t_0+a} \left(\frac{1}{2a} \right) f(t)\, dt \quad \left(\text{using the definition of } g(t) \text{ in } (1*) \right)$$

~~Tan~~ Taylor ~~expand~~ expand $f(t)$ near ~~ave~~ t_0,

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \lim_{a \to 0} \int_{t_0-a}^{t_0+a} \left(\frac{1}{2a} \right) \left(f(t_0) + \text{other terms that becomes zero as } a \to 0 \right) dt$$

$$= \lim_{a \to 0} \int_{t_0-a}^{t_0+a} \frac{1}{2a} f(t_0)\, dt$$

$$= \lim_{a \to 0} \frac{f(t_0)}{2a} \int_{t_0-a}^{t_0+a} 1 \, dt = \lim_{a \to 0} \frac{f(t_0)}{\cancel{2a}} \cdot \cancel{2a}$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)} \quad \text{(box 1)}$$

Example:

(i) $\int_{-\infty}^{\infty} \delta(t-5)\, e^{t}\, dt = e^{5}$

(ii) $\int_{-\infty}^{\infty} \delta(t)\, e^{t+2}\, dt = e^{2}$

L.T of Dirac delta function

let $a>0$,

$$L[\delta(t-a)] = \int_{0}^{\infty} \delta(t-a)\, e^{-st}\, dt \quad , \; s>0$$

$$= e^{-as}$$

(using the ~~definition~~ property of ~~of~~ in Dirac delta in box 1)

$$\boxed{L[\delta(t-a)] = \int_0^\infty \delta(t-a) e^{-st} = e^{-sa} \quad , \; \begin{matrix} s>0 \\ a>0 \end{matrix}}$$

Example: Solve the IVP using L.T.

$$y'' - y = -20\,\delta(t-3)$$

$$y(0) = 1, \quad y'(0) = 0$$

First, we transfor to frequency domain,

$$L[y''-y] = -20 L[\delta(t-3)]$$

$$L[y''] - L[y] = -20 L[\delta(t-3)] \quad \text{---} \; *1$$

$$L[y''] = s^2 Y(s) - s y(0) - y'(0)$$

$$L[\delta(t-3)] = \int_0^\infty \delta(t-3)\, e^{-st}\, dt = e^{-3s}$$

putting everything back into (*1).

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) - Y(s) = -20\, e^{-3s}$$

applying the initial conditions,

we have $y(0) = 1, \quad y'(0) = 0$

$$s^2 Y(s) - s - Y(s) = -20\, e^{-3s}$$

Simplifying,

$$Y(s) = \frac{-20\, e^{-3s} + s}{s^2 - 1}$$

(This is the solution of our IVP in frequency domain)

Let us find the inverse L.T. of Y(s)

$$Y(s) = \frac{-20\, e^{-3s}}{s^2 - 1} + \frac{s}{s^2 - 1}$$

$$y(t) = -20\, L^{-1}\left[\frac{e^{-3s}}{s^2-1}\right] + L^{-1}\left[\frac{s}{s^2-1}\right] \quad (**)$$

$$\frac{e^{-3s}}{s^2-1} = e^{-3s}\, L\left[\sinh(t)\right]$$

using formula 4 on our table,

$$L^{-1}\left[\frac{e^{-3s}}{s^2-1}\right] = u(t-3)\sinh(t-3)$$

using formula (20) with $a=0$ and $k=1$,

$$L\left[\cosh(t)\right] = \frac{s}{s^2-1}$$

$$\Rightarrow L^{-1}\left[\frac{s}{s^2-1}\right] = \cosh(t)$$

putting everything together in $(**)$,

$$y(t) = -20\, u(t-3)\sinh(t-3) + \cosh(t).$$

Webwork problem (Hint)

Given
$$y'' + y = \begin{cases} \sin(\pi t), & 0 \le t < 1 \\ 0, & 1 \le t \end{cases}$$

$$y(0) = 0, \quad y'(0) = 0$$

~~consider~~ Let

$$g(t) = \begin{cases} \sin(\pi t), & 0 \le t < 1 \\ 0, & 1 \le t \end{cases}$$

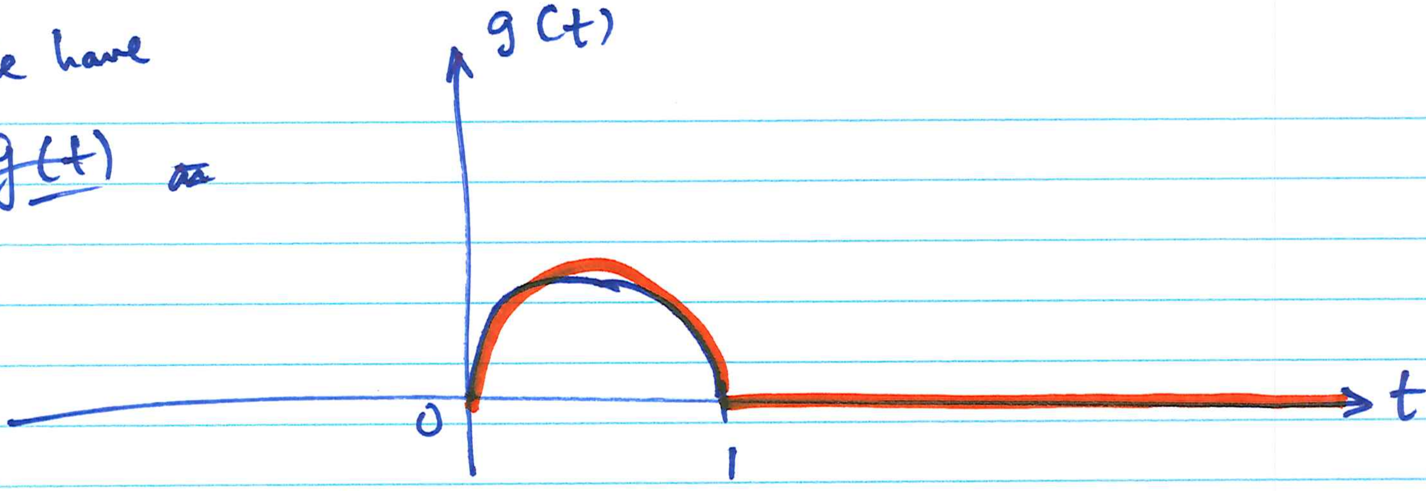
$$L[g(t)] = \int_0^\infty g(t)\, e^{-st}\, dt$$

$$= \int_0^1 \sin(\pi t)\, e^{-st}\, dt + \cancelto{0}{\int_1^\infty 0 \cdot e^{-st}\, dt}$$

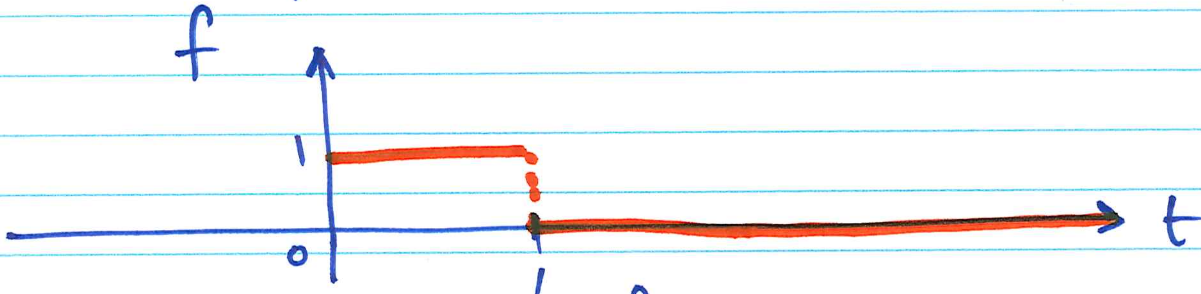
First method: Integration by parts twice.

Second method: write $g(t)$ in terms of unit step functions.

We have $g(t)$ as



construct $f = u(t) - u(t-1)$



$\therefore$ We can write $g(t)$ as follows,

$$g(t) = \left(u(t) - u(t-1)\right) \sin(\pi t)$$

$$g(t) = \underbrace{u(t)\sin(\pi t)} - \sin(\pi t)\, u(t-1)$$

(L.T. is straight forward using formula 4)

Consider

$\sin(\pi t)\, u(t-1)$ —— (**)

Let $k = t-1$, $t = k+1$

$$\Rightarrow \sin(\pi(k+1))\, u(k) = u(k)\left(\sin(\pi k)\cos(\pi) + \cancelto{0}{\cos(\pi k)\sin(\pi)}\right)$$

$$= u(k)\sin(\pi k)(-1)$$

$$(\cos(\pi) = -1)$$

$\therefore$ we have

$$-u(k)\sin(\pi k)$$

But $k = t-1$

$\therefore$ we have

$$-u(t-1)\sin(\pi(t-1))$$

$$\therefore g(t) = u(t)\sin(\pi t) + u(t-1)\sin(\pi(t-1))$$

$$L[g(t)] = L[u(t)\sin(\pi t)] + L[u(t-1)\sin(\pi(t-1))]$$

We ~~conte~~ can use formula 4 for the transforms.