Math215/255 Section 104 Quiz 3 (15 Minutes)

Name: Solution .... Student Number: ....

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Instructions: Answer ALL questions.

Question One:

Consider the following system of first order ODEs

$$\frac{dy_1}{dt} = 3y_1(t) - 4y_2(t)$$
$$\frac{dy_2}{dt} = y_1(t) - y_2(t)$$

(a) Find the general solution of the system.
(b) Use the initial conditions $y_1(0) = 1$ and $y_2(0) = 1$ to find the constants in your solution.
(c) Sketch the solution of the system near $(0, 0)$.
Show details of your solution.

Let $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

For the eigenvalues,

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = \lambda_2 = 1$

For the eigen vectors,

$(A - \lambda I)\vec{v}_1 = \vec{0}$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For ~~the second eigenvector~~ $\vec{v}_2$

$(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Let $u_2 = 0$, $u_1 = 1$ $\therefore \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$ The general solution is

$\vec{y}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] e^t$

To get the constants,

$\vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

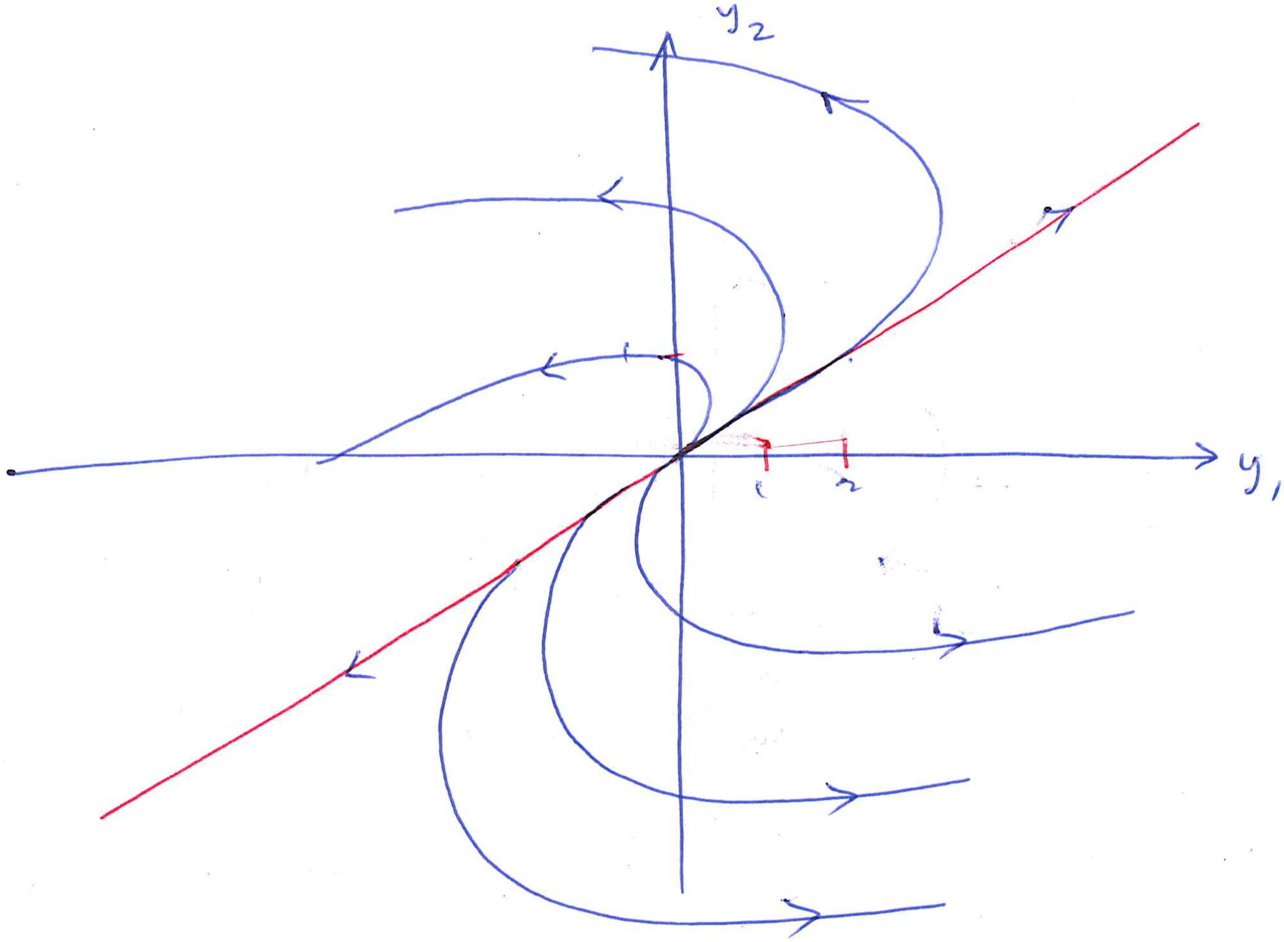
$\Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right]$

$c_1 = 1, \quad c_2 = -1$

$$\boxed{\vec{y}(t) = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + (-1)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] e^t}$$

$$\boxed{\vec{y}(t) = \begin{pmatrix}1\\1\end{pmatrix}e^{t} - t\begin{pmatrix}2\\1\end{pmatrix}e^{t}}$$



Question Two:

Consider the following system of first order ODEs

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_1(t) - 2y_2(t)$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = 3y_1(t) - 4y_2(t)$$

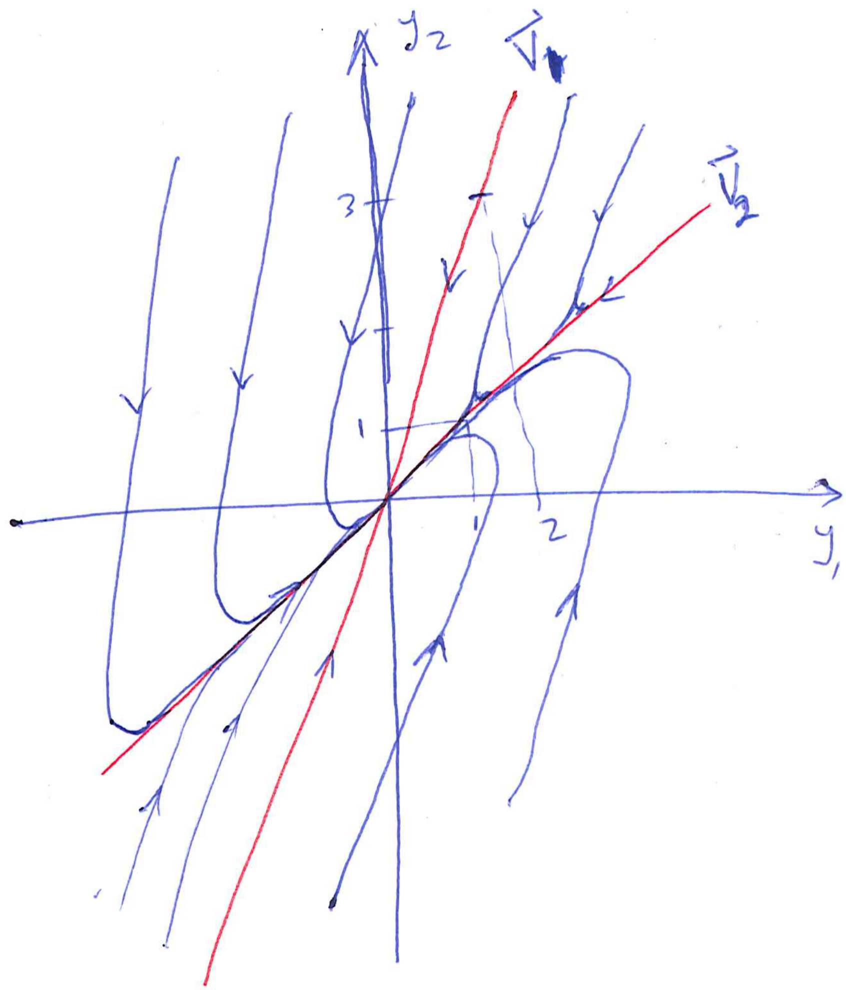
(a) Compute the eigenvalues and eigenvectors of the system.
(b) Find all equilibria (steady state solution) of the system and classify them.
(c) Use the eigenvalues and eigenvectors to sketch the solution of the system near $(0, 0)$.
Show details of your solution.

Let $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$

For the eigenvalues,

$\lambda^2 + 3\lambda + 2 = 0$

$\lambda_1 = -2$ and $\lambda_2 = -1$



For the eigenvectors,

$\lambda_1 = -2$,

$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{V}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda_2 = -1$, $(A - \lambda I)\vec{V}_2 = \vec{0}$

$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For the steady state solutions, we need to set

$$\vec{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and solve

$$\left[\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]$$

$$y_1 - 2y_2 = 0 \quad \Rightarrow \quad y_1 = 2y_2$$

$$3y_1 - 4y_2 = 0$$

$$\Rightarrow 3(2y_2) - 4y_2 = 0$$

$$6y_2 - 4y_2 = 0$$

$$2y_2 = 0$$

$$y_2 = 0$$

$$\Rightarrow y_1 = 0$$

$\therefore (y_1, y_2) = (0, 0)$ is the only equilibrium of the system.

It is a stable node.