Math215/255 Section 104 Quiz 4 (15 Minutes)

Name: Solution .... Student Number:..

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Instructions: Attempt ALL questions.

Question One:

Consider the forced system (Non-homogenoues system)

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_1(t) + y_2(t) + e^{-2t}$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = 4y_1(t) - 2y_2(t) - 2e^{t} \tag{1}$$

(a) Write the system in the form $\vec{Y}'(t) = A\vec{Y} + \vec{g}(t)$, where $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.

$$\vec{Y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{Y} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

(b) Find the fundament matrix for the homogeneous system $\vec{Y}'(t) = A\vec{Y}$.

$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$

For λ_1, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for λ_2, $\vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore$ The fundamental matrix is

$$\Psi = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix}$$

(c) Use the fundamental matrix to find ~~a~~ the general solution to the forced system in Equation 1.

$$\Psi^{-1} = \frac{1}{-5e^{-t}} \begin{pmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{pmatrix}$$

$$\Psi^{-1}\vec{g} = \frac{1}{5} \begin{pmatrix} 4e^{-4t} - 2e^{-t} \\ e^{t} + 2e^{4t} \end{pmatrix}$$

$$\int \Psi^{-1}\vec{g}\, dt = \frac{1}{5} \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ e^{t} + \frac{1}{2}e^{4t} \end{pmatrix}$$

$$\Psi \int \Psi^{-1}\vec{g}\, dt = \frac{1}{5} \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ e^{t} + \frac{1}{2}e^{4t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} \frac{5}{2}e^{t} \\ -5e^{-2t} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{t}$$

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{t}$$

(d) Use the initial condition $\overrightarrow{Y}(0) = \begin{pmatrix} \frac{5}{2} \\ -4 \end{pmatrix}$ to find the constants in your solution.

Applying the initial condition,

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-4\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix} = \begin{pmatrix}5/2\\-4\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\1 & -4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}5/2\\-4\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & 2\\1 & -4 & -3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 2\\0 & -5 & -5\end{array}\right]$$

$\Rightarrow c_2 = 1$, $\quad c_1 + c_2 = 2$

$c_1 = 2 - c_2 = 2 - 1 = 1$

$c_1 = 1$, and $c_2 = 1$ (This depends on how you have choosen your eigenvector)

BONUS

Suppose you are to solve the system in Equation 1 using the method of undetermined coefficient, what is the form of the particular solution $\overrightarrow{Y}_P$?

$$\overrightarrow{Y}_p(t) = \begin{pmatrix}a_1\\a_2\end{pmatrix}e^{-2t} + \begin{pmatrix}b_1\\b_2\end{pmatrix}e^{t}.$$