Last class

We started looking at the generic problem

$$y'' + by' + cy = e^{\alpha t} \quad \text{———} \quad (1)$$

we guessed $y_p = Ae^{\alpha t}$

put y_p into (1) to get

$$A = \frac{1}{\alpha^2 + b\alpha + c}$$

If α is a root of the characteristic polynomial $\lambda^2 + b\lambda + c = 0$

then A is undefined.

$\Rightarrow$ & our guess of $y_p = Ae^{\alpha t}$ has failed

Similar to the case of repeated roots, we multiply our guess by 't' to get a new guess.

$\therefore$ our new guess is

$$y_p = Ate^{\alpha t} \quad \text{———} \quad (2)$$

put ② into ①

$$y_p' = Ae^{\alpha t} + \alpha A t e^{\alpha t}.$$

$$y_p'' = \alpha A e^{\alpha t} + \alpha A e^{\alpha t} + \alpha^2 A t e^{\alpha t}$$

$$y_p'' = 2\alpha A e^{\alpha t} + \alpha^2 A t e^{\alpha t}.$$

$$\Rightarrow \left(2\alpha A e^{\alpha t} + \alpha^2 A t e^{\alpha t}\right) + b\left(Ae^{\alpha t} + \alpha A t e^{\alpha t}\right) + cAte^{\alpha t} = e^{\alpha t}.$$

Simplifying, we have

$$\left[(2\alpha A + bA) + (A\alpha^2 + bA\alpha + cA)t\right]e^{\alpha t} = e^{\alpha t}$$

But α is ~~a root of~~ $\lambda^2 + b\lambda + c = 0$

$$\Rightarrow A(\alpha^2 + b\alpha + c) = 0$$

$$(2\alpha + b)Ae^{\alpha t} = e^{\alpha t}$$

$$\Rightarrow A = \frac{1}{2\alpha + b} \Rightarrow y_p = \frac{t}{(2\alpha + b)} e^{\alpha t}$$

In general, if the form of the forcing function is already in our homogeneous solution, we need to multiply ~~th~~ our guess of the particular solution by t.

<u>Example</u>: Solve

$$y'' - 2y' - 3y = e^{-t}$$

Recall, $y_H = c_1 e^{3t} + c_2 e^{-t}$

Observe that $g(t) = e^{-t}$ and we already have e^{-t} in the homogeneous solution.

$\Rightarrow$ $y_p = Ae^{-t}$ will not work!!

$\therefore$ ~~A.~~ we multiply by t to get

$$y_p = Ate^{-t}$$

$$\Rightarrow \quad y_p' = Ae^{-t} \cancel{A} - Ate^{-t}$$

$$y_p'' = Ate^{-t} - 2Ae^{-t}.$$

put y_p into the ODE.

$$\left(Ate^{-t} - 2Ae^{-t}\right) - 2\left(Ae^{-t} - Ate^{-t}\right) - 3Ate^{-t}$$

$$= e^{-t}$$

Simplifying

$$(-2A - 2A)e^{-t} + (\overbrace{A + 2A\cancel{e^{-t}} - 3A}^{0})te^{-t} = e^{-t}$$

$$-4Ae^{-t} = e^{-t}$$

$$\Rightarrow \quad A = -1/4$$

$$\therefore \quad y_p = -1/4\, te^{-t}$$

$$\therefore \quad y(t) = c_1 e^{3t} + c_2 e^{-t} - 1/4\, te^{-t}$$

Suppose we are given $y(0) = 0$, $y'(0) = 0$

$y(0) = 0$

$$\Rightarrow \quad c_1 + c_2 = 0$$

$$\Rightarrow \quad c_1 = -c_2$$

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{1}{4} e^{-t} + \frac{1}{4} t e^{-t}$$

$y'(0) = 0$

$$\Rightarrow \quad 3c_1 - c_2 - \frac{1}{4} = 0$$

$$3c_1 + c_1 = \frac{1}{4}$$

$$4c_1 = \frac{1}{4} \quad \Rightarrow \quad c_1 = \frac{1}{16}$$

$$\Rightarrow \quad c_2 = -\frac{1}{16}$$

$$y(t) = \frac{1}{16} e^{3t} - \frac{1}{16} e^{-t} - \frac{1}{4} t e^{-t}$$

<u>Example:</u> Solve $y'' - 2y' - 3y = \sin(t)$.

We $y_H = c_1 e^{3t} + c_2 e^{-t}$

Guess:

$$y_p = A\sin(t) + B\cos(t).$$

$$y_p' = A\cos(t) - B\sin(t)$$

$$y_p'' = -A\sin(t) - B\cos(t)$$

put y_p into the ODE,

$$\left(-A\sin(t) - B\cos(t)\right) - 2\left(A\cos(t) - B\sin(t)\right)$$
$$-3\left(A\sin(t) + B\cos(t)\right) = \sin(t)$$

$$\left(-A + 2B - 3A\right)\sin(t) + \left(-B - 2A - 3B\right)\cos(t)$$
$$= \sin(t)$$

$\therefore$ collect coefficients,

$$-A + 2B - 3A = 1 \Rightarrow -4A + 2B = 1$$

$$-B - 2A - 3B = 0$$
$$2A = -4B$$
$$A = -2B$$

$$-4(-2B) + 2B = 1$$
$$\Rightarrow 10B = 1 \Rightarrow B = 1/10$$

$$A = -1/5$$

$$\therefore y_p = -\frac{1}{5}\sin(t) + \frac{1}{10}\cos(t).$$

$\therefore$ the general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{5}\sin(t) + \frac{1}{10}\cos(t)$$

Alternative method: "complexification"

We know that

$$e^{it} = \cos(t) + i\sin(t)$$

$$\Rightarrow \quad \sin(t) = \mathrm{Im}\left(e^{it}\right)$$

$\therefore$ we can write our ODE as

$$y'' - 2y' - 3y = \mathrm{Im}\left(e^{it}\right)$$

We shall solve for the particular solution of

$$y'' - 2y' - 3y = e^{it} \qquad *1$$

Guess: $y_p = Ae^{it}$

$y_p' = iAe^{it}$, $y_p'' = -Ae^{it}$

put y_p into (*1) ,

$$-Ae^{it} - 2iAe^{it} - 3Ae^{it} = e^{it}$$

$$\Rightarrow \quad (-4 - 2i)Ae^{it} = e^{it}$$

$$\Rightarrow \quad (-4 - 2i)A = 1$$

$$\Rightarrow \quad A = \frac{1}{-4 - 2i}$$

$$A = \frac{1}{(-4-2i)} \times \frac{(-4+2i)}{(-4+2i)} = \frac{-4+2i}{20} = -\frac{1}{5} + \frac{1}{10} i$$

$$\therefore y_p = Ae^{it} = \left(-\frac{1}{5} + \frac{1}{10} i\right) e^{it}$$

$$= \left(-\frac{1}{5} + \frac{1}{10} i\right)\left(\cos(t) + i\sin(t)\right)$$

$$y_p = \left(-\frac{1}{5}\cos(t) - \frac{1}{10}\sin(t)\right) + i\left(-\frac{1}{5}\sin(t) + \frac{1}{10}\cos(t)\right)$$

But we need $\mathrm{Im}(y_p)$.

$$\Rightarrow \mathrm{Im}(y_p) = -\frac{1}{5}\sin(t) + \frac{1}{10}\cos(t)$$

$\therefore y(t) = C_1 e^{3t} + C_2 e^{-t} - \frac{1}{5}\sin(t) + \frac{1}{10}\cos(t).$

Resonance and Beats

Consider the undamped forced system

$$y'' + w_0^2 y = g_0 \cos(wt)$$

or $g_0 \sin(wt)$

Resonance occurs when $w_0 = w$ (the ~~naturalral~~ forcing is done at the natural frequency of the system) while

Beats occurs when $w_0 \neq w$ (the forcing is done at a frequency different from the natural frequency of the system).

Resonance

Consider the classical example:

$$y'' + y = \cos(t)$$

For y_H, we solve

$$y_H'' + y_H = 0$$

$$\Rightarrow \quad \lambda^2 + 1 = 0$$

$$\Rightarrow \quad \lambda = \pm i$$

$$\therefore \quad y_H = C_1 \cos(t) + C_2 \sin(t)$$