Example: solve

$$\vec{y}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{y}(t)$$

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

Let λ be an eigenvalue of A,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2 \text{ (twice)}$$

For the eigenvectors,

$$(A - \lambda I)\vec{V}_1 = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\vec{V}_2$, we solve

$$(A - \lambda I)\vec{V}_2 = \vec{V}_1$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow$ $-u_1 - u_2 = 1$, $-u_1 = 1 + u_2$

$u_1 = -1 - u_2$ $\qquad$ $u_1 = -u_2 - 1$

take $u_2 = 0$, $\Rightarrow$ $u_1 = -1$

$$\vec{V}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$\therefore$ The general solution is

$$\vec{Y}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]e^{2t}$$

Matlab Demo

$$\vec{Y}'(t) = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{Y}(t)$$

with $\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution is

$$\vec{Y}(t) = \frac{2}{5} e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{5} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

In component form,

$$\vec{Y}_1(t) = \frac{4}{5} e^{4t} + \frac{1}{5} e^{-t}$$

$$\vec{Y}_2(t) = \frac{6}{5} e^{4t} - \frac{1}{5} e^{-t}$$

Review of complex numbers

A complex number is any number of the form

$$z = x + iy$$

where x and y are real numbers

and $i = \sqrt{-1}$

$x = \mathrm{Re}(z)$, real part of z

$y = \mathrm{Im}(z)$, imaginary part of z

Let $z_1 = 4 + 5i$ and $z_2 = 3 + 2i$

$$\begin{aligned} z_1 + z_2 &= (4+5i) + (3+2i) \\ &= (4+3) + i(5+2) \\ &= 7 + 7i \end{aligned}$$

$$z_1 \times z_2 = (4+5i) \times (3+2i)$$
$$= 12 + 8i + 15i + 10i^2$$

but $i^2 = -1$

$$\Rightarrow z_1 \times z_2 = (12-10) + i(8+15)$$
$$= \underline{\underline{2 + 23i}}$$

<u>complex conjugate</u>

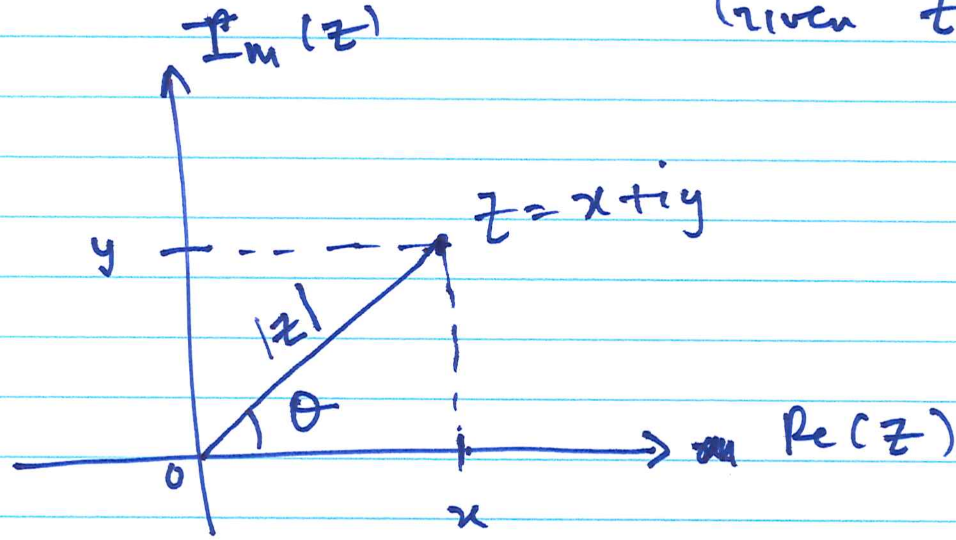
Let $z = x+iy$, the complex conjugate of z is $\bar{z} = x - iy$

<u>complex division</u>

$$\frac{z_1}{z_2} = \frac{(4+5i)\times(3-2i)}{(3+2i)\times(3-2i)} = \frac{22}{13} + \frac{7}{13}i$$

Argand diagram

Given $z = x + iy$



Modulus of z, $|z| = \sqrt{x^2 + y^2}$

argument of z, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

SOH CAH TOA, let $r = |z|$

$\sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = x/r \Rightarrow x = r\cos\theta$

But $z = x + iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

Consider the Taylor expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{it} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots$$

$$= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)}_{\cos(t)} + i\underbrace{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)}_{\sin(t)}$$

$$\Rightarrow \boxed{e^{it} = \cos(t) + i\sin t}$$ (Euler's formula)

Return to z,

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\boxed{z = re^{i\theta}}$$ (polar representation of z)

$$e^{i\theta} + e^{-i\theta} = \left(\cos(\theta) + i\sin\theta\right) + \left(\cos(\theta) - i\sin\theta\right)$$

$$= 2\cos(\theta)$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Similarly,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

SYSTEMS WITH COMPLEX EIGENVALUES

Example: Find the general solution of the system

$$\vec{y}\,'(t) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \vec{y}(t)$$

Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

If λ is an eigenvalue of A,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda + 5 = 0$$

using the quadratic formula

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm 2i$$

$\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$

For the eigenvectors,

$$(A-\lambda_1 I)\vec{V}_1=\vec{0}$$

$$\Rightarrow \left(\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}-(1+2i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-2i & -2\\ 2 & -2i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Rightarrow \vec{V}_1=\begin{pmatrix}i\\ 1\end{pmatrix}$$

For $\vec{V}_2$, $(A-\lambda_2 I)\vec{V}_2=\vec{0}$

$$\left(\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}-(1-2i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}2i & -2\\ 2 & 2i\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\vec{V}_2=\begin{pmatrix}-i\\ 1\end{pmatrix}$$