CLASSIFICATION OF FIXED POINTS / EQUILIBRIUM SOLUTION / Steady-state solution for 2 dimensional system

For constant-coefficient linear systems of the form

$$\vec{Y}'(t) = \vec{F}(\vec{Y}(t)) = A\vec{Y}(t).$$

$(y_1, y_2) \equiv (0,0)$ is alway an equilibrium solution.

We shall classify this equilibrium solution and sketch the solution of the system close to this point.

Case I : Real distinct eigenvalues.

(a) $\lambda_1, \lambda_2 > 0$

Example: $y_1'(t) = 2y_1 + y_2$

$y_2'(t) = y_1 + 2y_2$

For steady-state solution, set $y_1'(t) = 0$, $y_2'(t) = 0$

$2y_1 + y_2 = 0$

$y_1 + 2y_2 = 0$

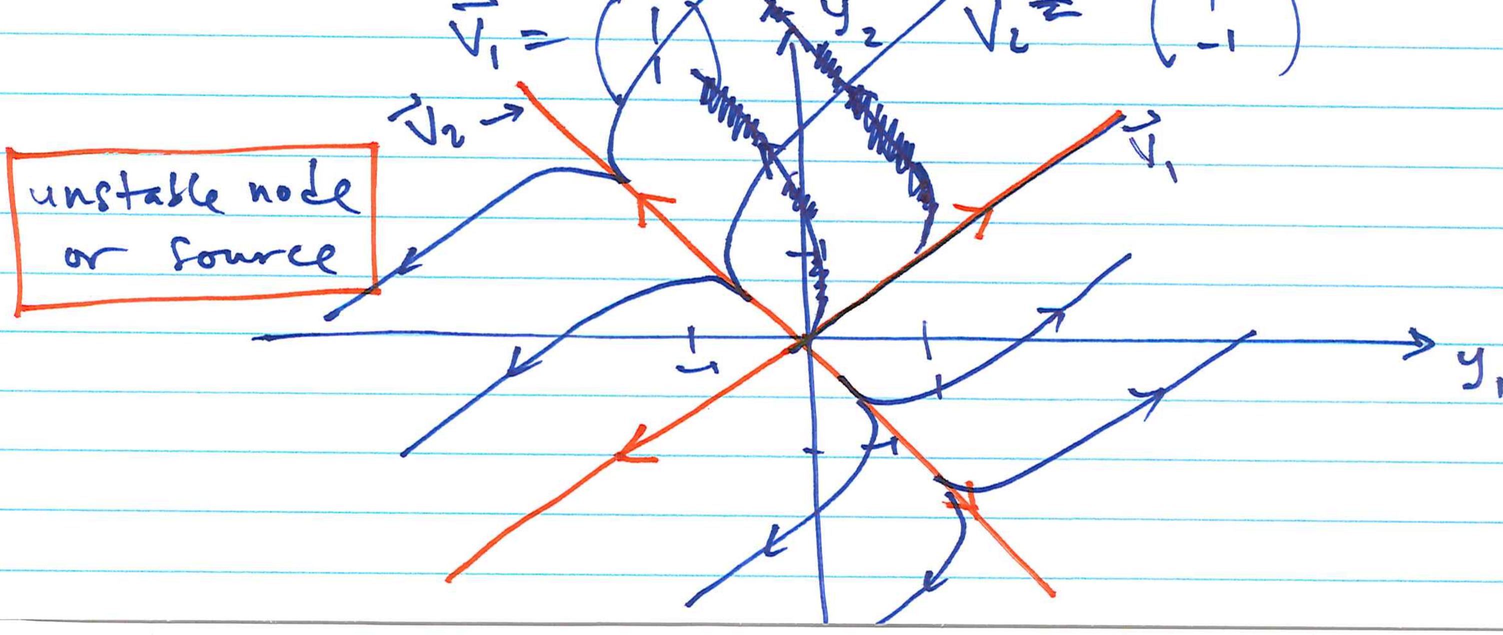
$\Rightarrow \quad y_1 = 0 \ , \ y_2 = 0$

$\vec{y}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the steady-state

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$

$\vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

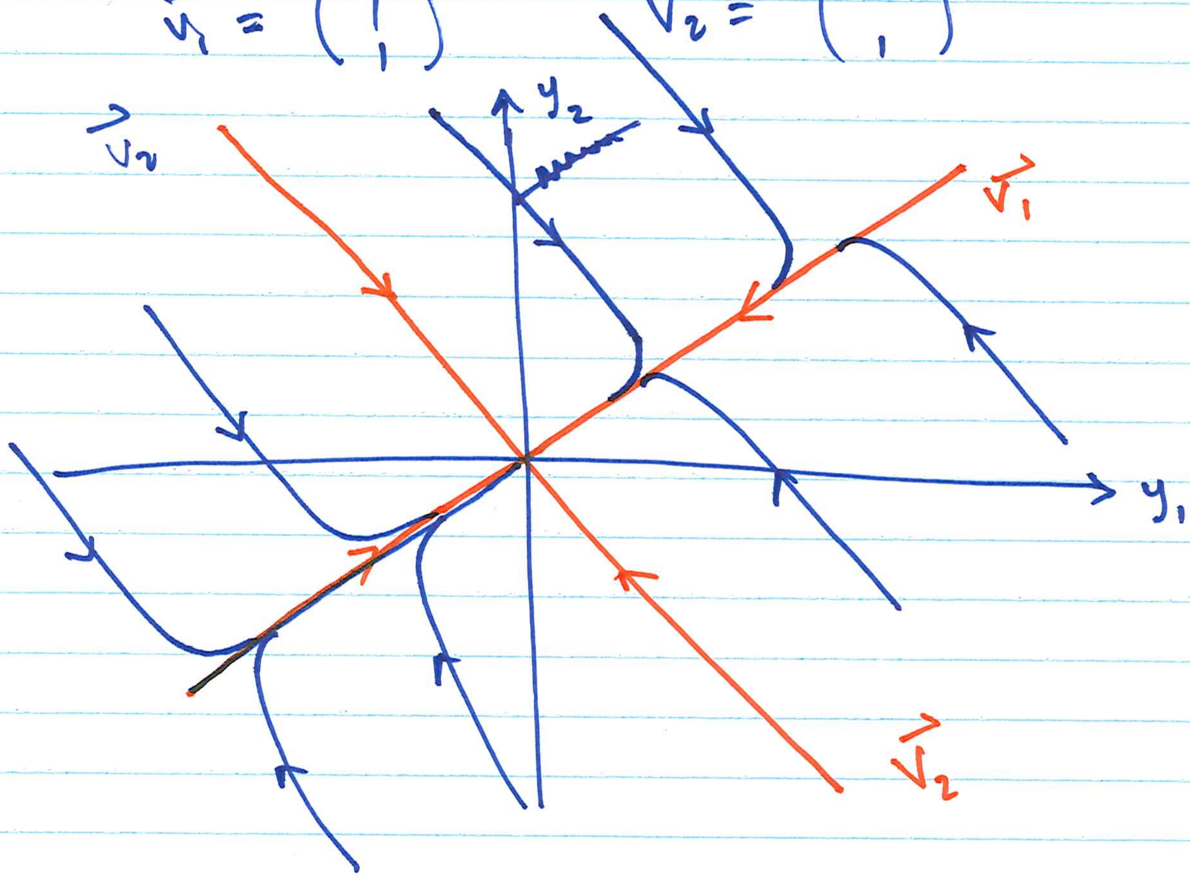


(b) $\lambda_1, \lambda_2 < 0$

Example: $\vec{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = -1$ and $\lambda_2 = -3$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



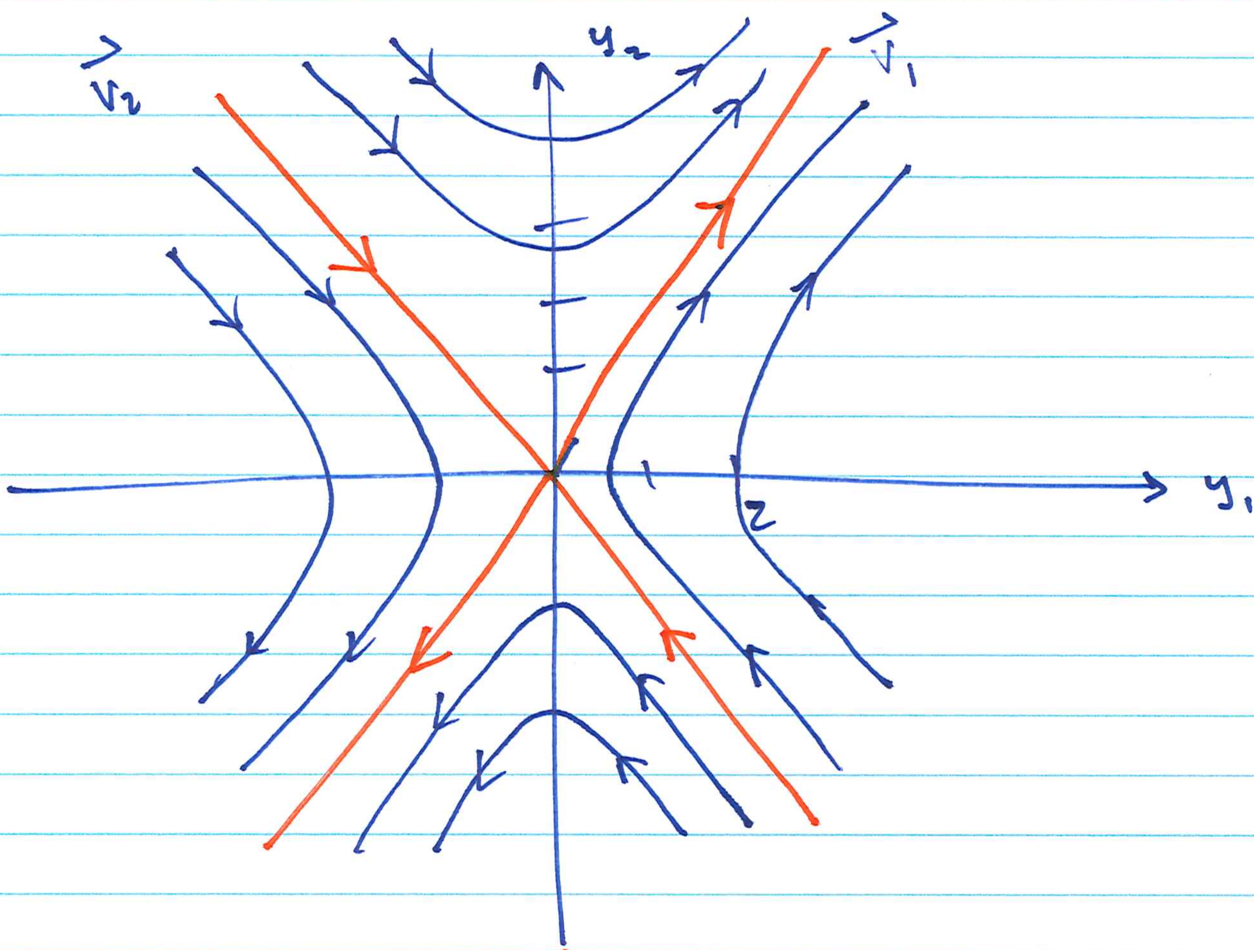
$\boxed{\text{Stable node / Sink}}$

(c) $\lambda_1 > 0$ and $\lambda_2 < 0$

Example: $\vec{y}'(t) = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = 4$ $\qquad\qquad$ $\lambda_2 = -1$

$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\qquad\qquad$ $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Saddle

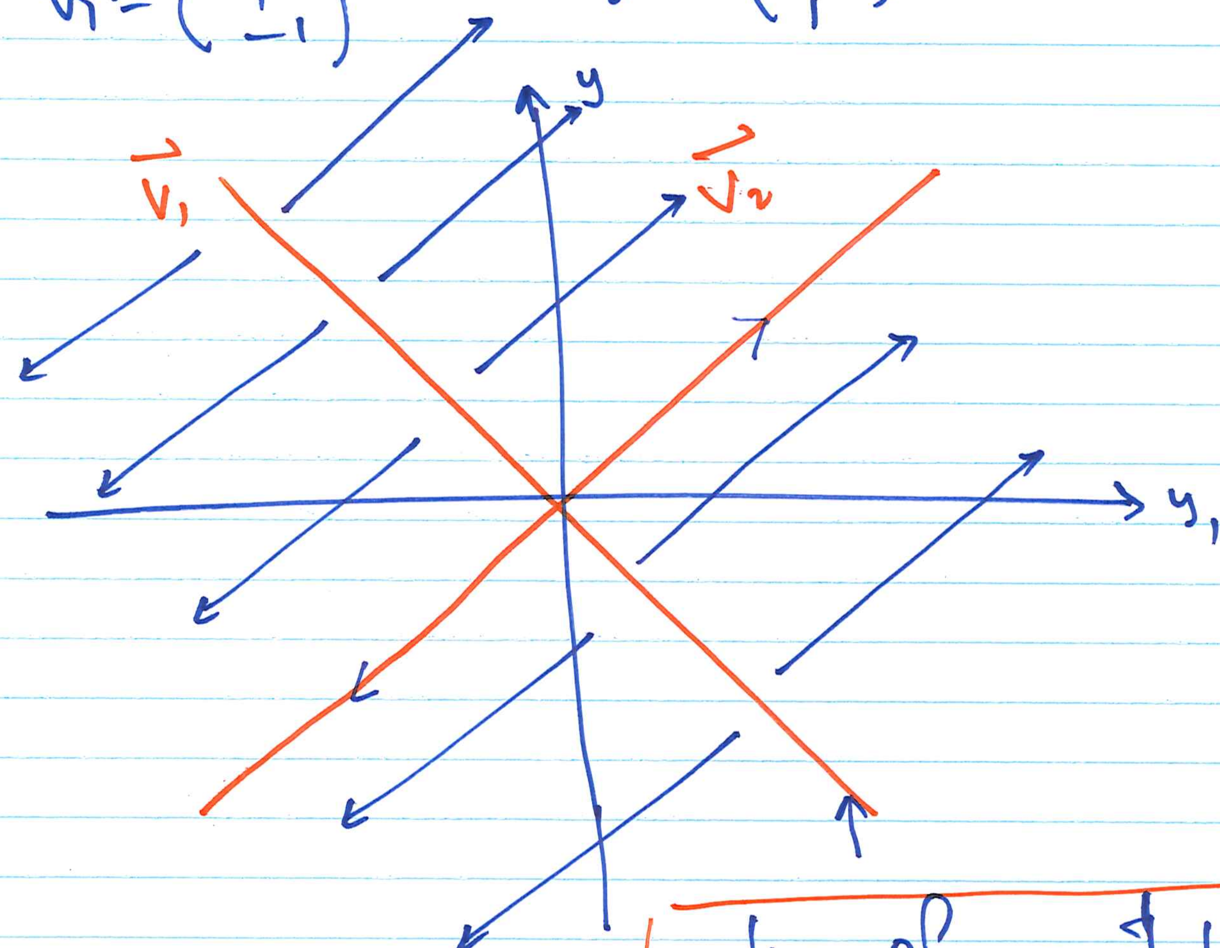
unstable except the initial condition is on $\vec{v}_2$.

(d) $\lambda_1 = 0$, $\lambda_2 > 0$

$\vec{y}'(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = 0$, $\lambda_2 = 2$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



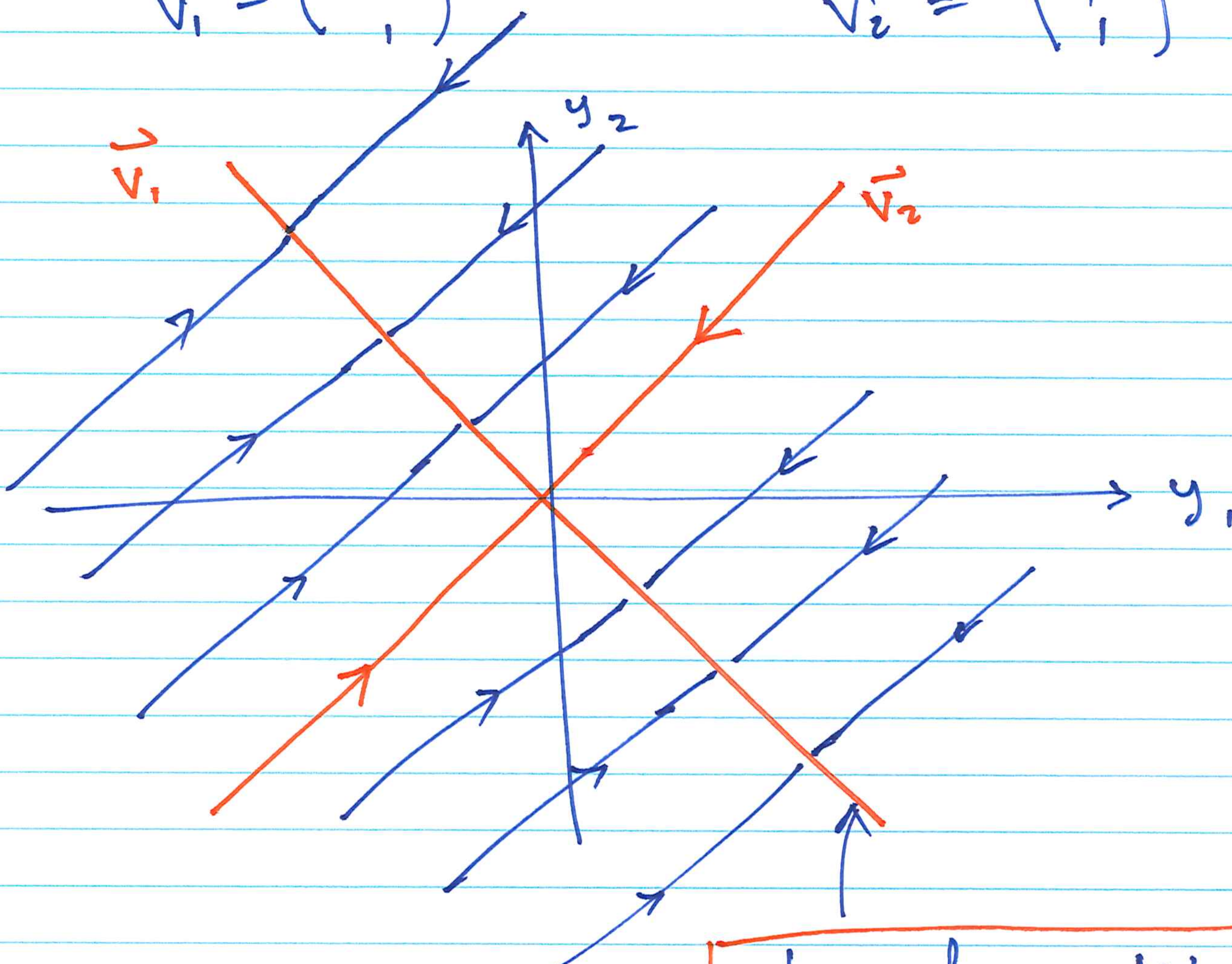
$\boxed{\text{line of equilibria and it's unstable}}$

(e) $\lambda_1 = 0, \quad \lambda_2 < 0.$

$$\vec{y}'(t) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \vec{y}(t)$$

$\lambda_1 = 0$ $\qquad\qquad$ $\lambda_2 = -2$

$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\qquad\qquad$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



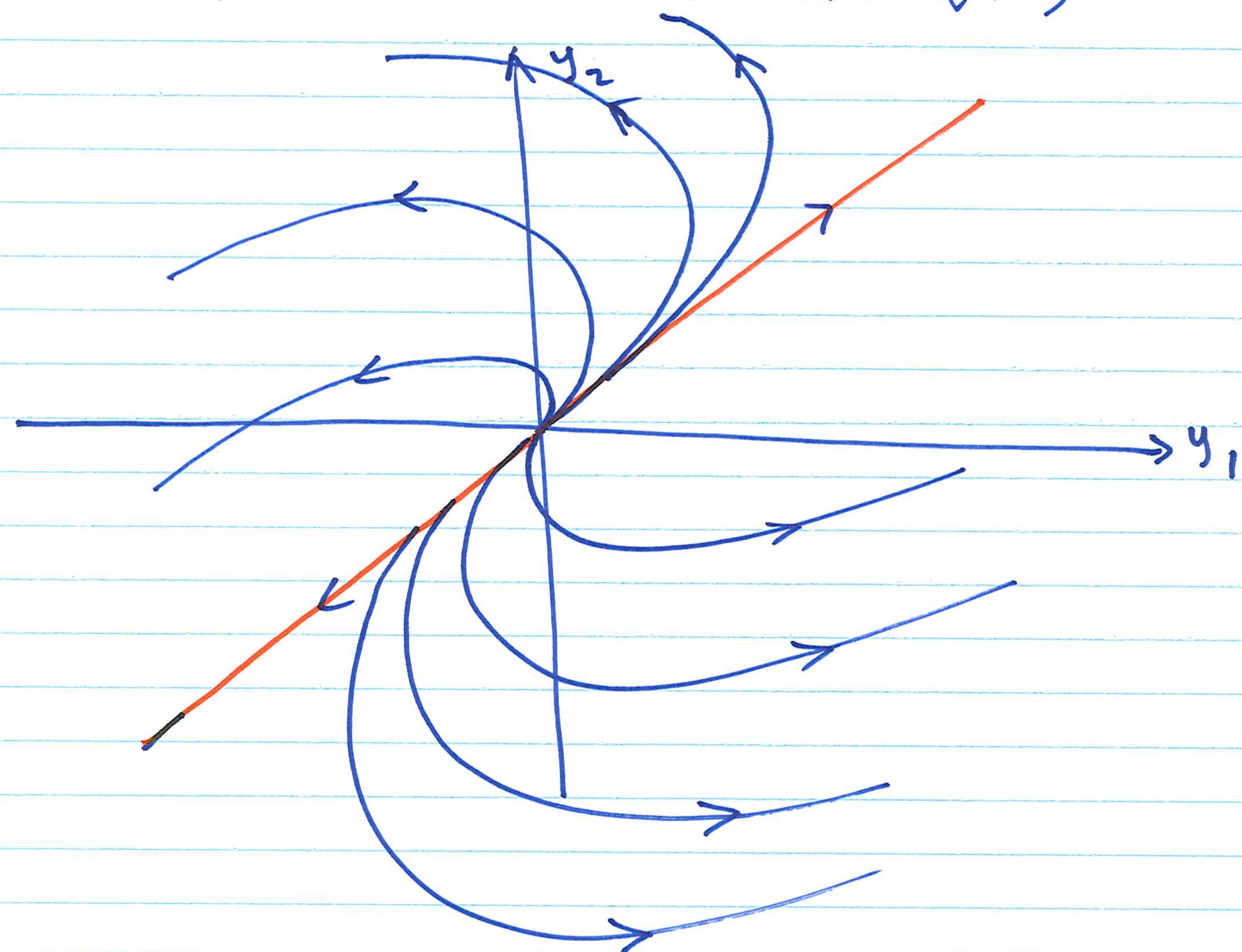
line of equilibria and it's stable

Case II : Real repeated eigenvalues

(a) $\lambda_1 = \lambda_2 > 0$

Example: $\vec{y}\,'(t) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = \lambda_2 = 1$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

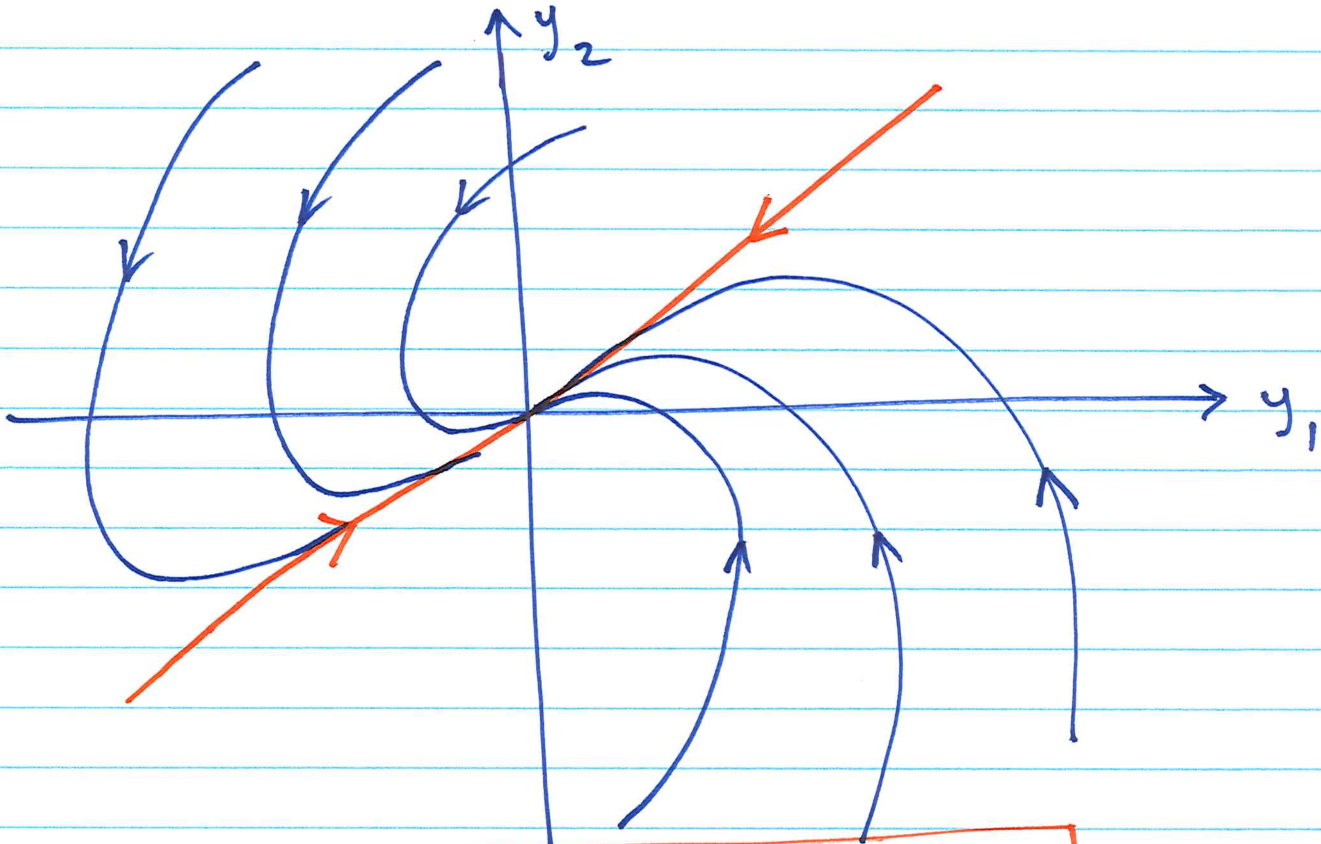


$\boxed{\text{Improper node and unstable}}$

(6) $\lambda_1 = \lambda_2 < 0$

$$\vec{y}'(t) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \vec{y}(t)$$

$\lambda_1 = \lambda_2 = -2$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



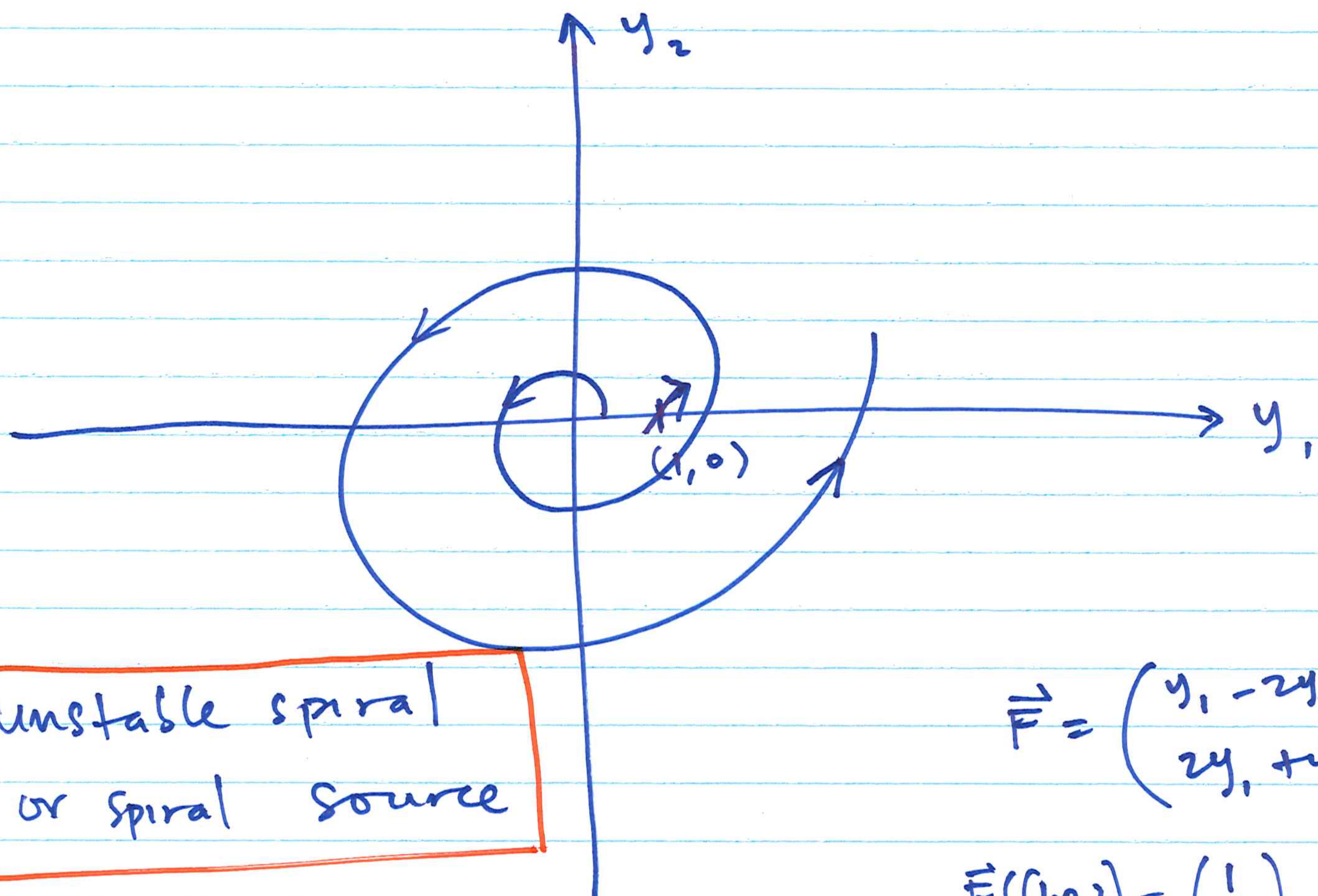
$\boxed{\text{Improper node and stable or stable improper node}}$

case III : complex eigenvalues.

$\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.

(a) $\alpha > 0$, $\vec{y}'(t) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \vec{y}(t)$

$\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$



$\boxed{\text{unstable spiral or spiral source}}$

$\vec{F} = \begin{pmatrix} y_1 - 2y_2 \\ 2y_1 + y_2 \end{pmatrix}$

$\vec{F}((1,0)) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$