Separable 1st order ODEs

Examples: Solve

$$\frac{dy}{dt} = \frac{(t^2+1)}{y}, \quad y(0) = 1$$

$$\int y \, dy = \int (t^2+1)\, dt + C_1$$

$$\frac{y^2}{2} = \frac{t^3}{3} + t + C_1$$

$$y = \pm\sqrt{\frac{2t^3}{3} + 2t + 2C_1}$$

$$y = \pm\sqrt{\frac{2t^3}{3} + 2t + C_2} \qquad (C_2 = 2C_1)$$

$$y(0) = 1 \Rightarrow 1 = \pm\sqrt{C_2}$$

$$\Rightarrow C_2 = 1$$

$$y = \sqrt{\frac{2t^3}{3} + 2t + 1}$$

In general, given an ODE of the form

$$\frac{dy}{dt} + \overbrace{M(t)\,N(y)}^{f(t,y)} = 0$$

$$\frac{dy}{dt} = -M(t)\,N(y)$$

$$\int \frac{dy}{N(y)} = \int -M(t)\,dt + C$$

then we write the solution as

$y(x) = \text{— —} \ldots$

Example: Solve the IVP

$$x \, dy/dx = y + 2x^2 y \ , \quad y(1) = 1$$

$$x \, dy/dx = y(1 + 2x^2)$$

$$\int \frac{dy}{y} = \int \frac{(1+2x^2)}{x} dx + C_1$$

$$\ln y = \ln x + x^2 + C_1$$

$$y = e^{\ln x} \cdot e^{x^2} \cdot e^{C_1} \quad \left(C_2 = e^{C_1} \right)$$

$$y = C_2 \, x \, e^{x^2}$$

$y(1) = 1$

$$\Rightarrow \quad 1 = C_2 \cdot 1 \cdot e^1 \quad \Rightarrow \quad C_2 = e^{-1}$$

$$\therefore y = x \, e^{x^2 - 1}$$

Integrating factor method

Example: Solve

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}, \quad y(1) = 0$$

~~Take~~ Multiply through by x^3

$$x^3 \frac{dy}{dx} + 3x^2 y = e^x$$

Observe that

$$\frac{d}{dx}\left(x^3 y\right) = x^3 y' + 3x^2 y$$

$$\Rightarrow \quad \frac{d}{dx}(x^3 y) = e^x$$

$$\int d(x^3 y) = \int e^x \, dx + C_1$$

$$x^3 y = e^x + C_1$$

$$y = \frac{e^x}{x^3} + \frac{C_1}{x^3}$$

$y(1) = 0$

$$\Rightarrow \quad 0 = \frac{e^1}{1} + \frac{c_1}{1}$$

$$\Rightarrow \quad c_1 = -e^1$$

$$\therefore \quad y = \frac{1}{x^3}\left(e^x - e^1\right)$$

Que: How did I get the factor 'x^3'?

Let us look at the general case

Given

$$a(t)\,y'(t) + b(t)\,y(t) = c(t)$$

divide through by $a(t)$ and set

$$p(t) = \frac{b(t)}{a(t)}, \quad q(t) = \frac{c(t)}{a(t)}$$

$$y'(t) + p(t)\,y(t) = \varphi(t) \quad (1)$$

We want to write the L.H.S as

$\frac{d}{dt}\left(h(t)\,y(t)\right)$ for some function $h(t)$.

Multiply eq (1) by $h(t)$

$$h(t)\,y'(t) + h(t)\,p(t)\,y = h(t)\,\varphi(t) \quad (2)$$

But

$$\frac{d}{dt}\left(h(t)\,y(t)\right) = h(t)\,y'(t) + h'(t)\,y(t) \quad (3)$$

compare (3) with L.H.S of (2)

$\Rightarrow$ $h'(t) = h(t)\,p(t)$ must hold

$$\frac{dh(t)}{dt} = h(t)\,p(t)$$

$$\int \frac{dh(t)}{h(t)} = \int p(t)\,dt + c_1$$

$$\ln h(t) = \int p(t)\,dt + c_1$$

$$h(t) = e^{\int p(t)\,dt} \cdot e^{c_1}$$

$$h(t) = c_2\, e^{\int p(t)\,dt} \quad \left(c_2 = e^{c_1}\right)$$

$\boxed{\text{take } c_2 = 1}$

$$h(t) = e^{\int p(t)\,dt}$$

Substitute $h(t)$ in (2)

$$e^{\int p(t)\,dt}\, y' + e^{\int p(t)\,dt}\, p(t)\, y = e^{\int p(t)\,dt}\, \varphi(t)$$

$$\frac{d}{dt}\left[e^{\int p(t)\,dt}\, y \right] = e^{\int p(t)\,dt}\, \varphi(t)$$

Integrating,

$$e^{\int p(t)\,dt}\, y(t) = \int e^{\int p(t)\,dt}\, \varphi(t) + C_3$$

$$\boxed{y(t) = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt}\, \varphi(t) + C_3\, e^{-\int p\,dt}}$$

Remark

The function $h(t) = e^{\int p(t)\,dt}$ is called the integrating factor.

Return to the previous example:

$$dy/dx + 3y/x = \frac{e^x}{x^3}$$

$$p(x) = 3/x \ , \ \varphi(x) = \frac{e^x}{x^3}$$

$$h(x) = e^{\int p(x)\,dx}$$

$$= e^{\int 3/x\,dx} = e^{3\int 1/x\,dx}$$

$$= e^{\ln x^3} = x^3$$

$$\Rightarrow h(x) = x^3$$

Example: Solve $dy/dt + 2ty = t.$

$p(t) = 2t \quad , \varphi(t) = t$

$$h(t) = e^{\int p(t)\,dt} = e^{\int 2t\,dt} = e^{t^2}$$

$$e^{t^2}\,dy/dt + 2te^{t^2}y = te^{t^2}$$

$$\Rightarrow \frac{d}{dt}\left(e^{t^2}y\right) = te^{t^2}$$

$$\int d\left(e^{t^2} y\right) = \int t e^{t^2} dt + c_1$$

$$e^{t^2} y = \frac{1}{2} e^{t^2} + c_1$$

$$y = \frac{1}{2} + c_1 e^{-t^2}$$

Details of the integral

$$\int t e^{t^2} dt = \frac{1}{2} \int 2t e^{t^2} dt$$

Let $u = t^2$, $du = 2t\,dt$

$$\int t e^{t^2} dt = \frac{1}{2} \int e^u du = \frac{e^u}{2}$$

$$= \frac{e^{t^2}}{2}$$